Other forms of continuity modulo an ideal

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Abstract:

The topic analized in this paper is the continuity modulo an ideal. We use the open-I sets to introduce new forms of weak continuity. Some basic properties of C-continuous and D-continuous functions will be investigated, as well as some applications related to compactness and separability. All the results obtained in this work constitute generalizations of well-known results of the general topology. We prove that these new concepts are independent of other forms of weak continuity, modulo an ideal, introduced by Abd El-Monsef, Özkurt, Çobankaya and Kaniewski.

1. Introduction and preliminaries

Many concepts of the general topology have been extended to ideal topological spaces, and it is the case that the same concept has several extensions. The case of continuity is a good example of this fact. Authors such as Abd El-Monsef, Kaniewski, Özkurt and Çobankaya have studied different versions of weak continuity, modulo an ideal. In this paper we present two new forms of continuity for ideal topological spaces, which are not related to the versions presented by those authors.

An ideal $\mathcal{I}$ in a set X is a subset of $\mathcal{P}(X)$, the power set of X, such that: (i) if $A \subseteq B \subseteq X$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$, and (ii) if $A \in \mathcal{I}$ and $B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.

Some simple and useful ideals in X are: $(i)\mathcal{P}(A)$, where $A \subseteq X$, $(ii)\mathcal{I}_f(X)$, the ideal of all finite subsets of X and (iii) $\mathcal{I}_c(X)$, the ideal of all countable subsets of X. If $\mathcal{I}$ is an ideal in X and if $f : X \to Y$ is a function, then the set $f(\mathcal{I}) = \{f(I) : I \in \mathcal{I}\}$ is an ideal in Y [7]. Furthermore, if $\mathcal{J}$ is an ideal in Y and if $f : X \to Y$ is an one-one function, then the set $f^{-1}(\mathcal{J}) = \{f^{-1}(J) : J \in \mathcal{J}\}$ is an ideal in X [7]. An ideal $\mathcal{I}$ in X is said to be admissible if $\{x\} \in \mathcal{I}$, for each $x \in X$.

If (X,τ) is a topological space and $\mathcal{I}$ is an ideal in X, then $(X,\tau,\mathcal{I})$ is called an ideal space. If (X,τ) is a topological space and $A \subseteq X$ then the closure and the interior of A are denoted by $\overline{A}$ (or $adh_\tau(A)$, or $adh(A)$) and $\mathring{A}$ (or $int_\tau(A)$, or $int(A)$), respectively. The frontier of A is denoted by $Fr(A)$.

Given an ideal space $(X,\tau,\mathcal{I})$ and a set $A \subseteq X$, we denote by $A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I}$, for every $U \in \tau$ with $x \in U\}$, written simply as A^* when there is no chance for confusion. It is clear that $A^* \subseteq \overline{A}$. A Kuratowski closure operator for a topology $\tau^*(\mathcal{I})$, finer than τ, is defined by $Cl^*(A) = A \cup A^*$, for all $A \subseteq X$. When there is no chance for confusion $\tau^*(\mathcal{I})$ is denoted by τ^*. The topology τ^* has as a base $\beta(\tau,\mathcal{I}) = \{V \backslash I : V \in \tau$ and $I \in \mathcal{I}\}$ [5]. In 1990, D. Jancovic and T. R. Hamlett introduced the notion of $\mathcal{I}$-open sets. If $(X,\tau,\mathcal{I})$ is an ideal space and $A \subseteq X$, A is said to be $\mathcal{I}$-open [4] if $A \subseteq int(A^*)$. A is said to be $\mathcal{I}$-closed if $X\backslash A$ is $\mathcal{I}$-open. On the other hand, A is said to be closed-$\mathcal{I}$ [10] if $\overline{A}\backslash A \in \mathcal{I}$. If $X\backslash A$ is closed-$\mathcal{I}$ then A is defined to be open-$\mathcal{I}$. It is immediate that A is open-$\mathcal{I}$ if and only if $A\backslash \mathring{A} \in \mathcal{I}$.

If $\mathcal{I}$ is an ideal in X, let $\mathcal{I}^{\otimes} = \mathcal{P}\left(\bigcup_{I \in \mathcal{I}} I\right)$ be. If $(X,\tau,\mathcal{I})$ is an ideal space, the ideal $\overline{\mathcal{I}}$ [10] is the set $\left\{A \subseteq X : A \subseteq \overline{I}, \text{ for some } I \in \mathcal{I}\right\}$.

It is observed that if $\{F_\alpha\}_{\alpha\in\Lambda}$ is a locally finite collection of closed-$\mathcal{I}$ sets, then $\bigcup\limits_{\alpha\in\Lambda} F_\alpha$ is closed-$\mathcal{I}^\otimes$. In fact, since $\{F_\alpha\}_{\alpha\in\Lambda}$ is locally finite we have that $\overline{\bigcup\limits_{\alpha\in\Lambda} F_\alpha} = \bigcup\limits_{\alpha\in\Lambda} \overline{F_\alpha}$, and so $\overline{\bigcup\limits_{\alpha\in\Lambda} F_\alpha} \backslash \bigcup\limits_{\alpha\in\Lambda} F_\alpha = \left(\bigcup\limits_{\alpha\in\Lambda} \overline{F_\alpha}\right) \backslash \left(\bigcup\limits_{\alpha\in\Lambda} F_\alpha\right) \subseteq$ $\bigcup\limits_{\alpha\in\Lambda} \left(\overline{F_\alpha}\backslash F_\alpha\right) \in \mathcal{I}^\otimes$.

2. $\mathcal{C}$-continuous functions

In 2014 Özkurt defines the $\mathcal{J}$-continuous functions, as an extension of the continuous functions to the ideal topological spaces. Years later, in 2017, Çobankaya et al. defines the $\mathcal{J}_w$-continuous functions, as a generalization of the concept studied by Özkurt. In both cases the considered ideal is defined in the codomain of the function. Under the same hypothesis, in this section we define a new and natural extension of the continuous functions to the ideal topological spaces, which turns out to be independent of the concepts given by these two authors. Various properties and examples will be presented.

Recall that if (X,τ) and (Y,β) are topological spaces and $\mathcal{J}$ is an ideal in Y, a function $f : (X,\tau) \to (Y,\beta,\mathcal{J})$ is said to be $\mathcal{J}$-continuous [8] if for each $x \in X$ and each $V \in \beta$, if $f(x) \in V$ then there exists $U \in \tau$ such that $x \in U$ and $f(U)\backslash V \in \mathcal{J}$. A function $f : (X,\tau) \to (Y,\beta,\mathcal{J})$ is said to be $\mathcal{J}_w-continuous$[2] if for each $x \in X$ and each $V \in \beta$, if $f(x) \in V$ then there is a $U \in \tau$ such that $x \in U$ and $f(U)\backslash \overline{V} \in \mathcal{J}$. Evidently $\mathcal{J}$-continuous $\to \mathcal{J}_w$-continuous.

Definition 2.1 If $\mathcal{J}$ is an ideal in Y, a function $f : (X,\tau) \to (Y,\beta,\mathcal{J})$ is said to be $\mathcal{C}$-continuous if, for each $A \subseteq X$, $f\left(\overline{A}\right)\backslash\overline{f(A)} \in \mathcal{J}$.

It is clear that if f is continuous then f is $\mathcal{C}$-continuous, and that $f : (X,\tau) \to (Y,\beta)$ is continuous if and only if $f : (X,\tau) \to (Y,\beta,\{\emptyset\})$ is $\mathcal{C}$-continuous. It is easy to see that if $g : (X,\tau) \to (Y,\beta)$ is continuous and $h : (Y,\beta) \to (Z,\gamma,\mathcal{J})$ is $\mathcal{C}$-continuous, then $h \circ g$ is $\mathcal{C}$-continuous. Also it is immediate that $f : (X,\tau) \to (Y,\beta,\mathcal{J}_\alpha)$ is $\mathcal{C}$-continuous, for each $\alpha \in \Lambda$, if and only if $f : (X,\tau) \to (Y,\beta,\mathcal{J})$ is $\mathcal{C}$-continuous, where $\mathcal{J} = \bigcap\limits_{\alpha\in\Lambda} \mathcal{J}_\alpha$.

Example 2.2 (1) If $\mathcal{U}$ is the usual topology in $\mathbf{R}$, the identity function $f : (\mathbf{R},\mathcal{U}) \to (\mathbf{R},\mathcal{P}(\mathbf{R}),\mathcal{J} = \mathcal{P}(\mathbf{R}))$ is $\mathcal{C}$-continuous, but f is not continuous.
(2) If $\beta = \{\emptyset,\mathbf{R},\{0\}\}$ and $\mathcal{J} = \mathcal{P}(\{-1,0,1\})$, the function $f : (\mathbf{R},\mathcal{U}) \to (\mathbf{R},\beta,\mathcal{J})$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is $\mathcal{C}$-continuous but f is not continuous.

(3) The function $f : (\mathbf{R}, \mathcal{U}) \to (\mathbf{R}, \mathcal{U}, \mathcal{J} = \mathcal{P}(\{0\}))$ defined by $f(x) = [x]$, where $[x]$ denotes the integer part of x, is not $\mathcal{C}$-continuous, because if $A = \{1 - 1/n : n \in \mathbf{Z}^+\}$ then $f\left(\overline{A}\right) \setminus \overline{f(A)} = \{1\} \notin \mathcal{J}$.

Next we present a characterization of the $\mathcal{C}$-continuity in terms of the interior operator.

Theorem 2.3 The function $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is $\mathcal{C}$-continuous if and only if, for each $B \subseteq Y$, there exists $J \in \mathcal{J}$ such that $f^{-1}(B^\circ \setminus J) \subseteq int(f^{-1}(B))$ or, equivalently, $f^{-1}\left(\overset{\circ}{B}\right) \subseteq int(f^{-1}(B)) \cup f^{-1}(J)$. Then f is $\mathcal{C}$-continuous if and only if, for all $B \subseteq Y$, $f[f^{-1}(B^\circ) \setminus int(f^{-1}(B))] \in \mathcal{J}$.

Proof. ($\to$) Suppose that $B \subseteq Y$. There exists $J \in \mathcal{J}$ with $f\left[\overline{f^{-1}(Y \setminus B)}\right] \subseteq \overline{f[f^{-1}(Y \setminus B)]} \cup J \subseteq \overline{(Y \setminus B)} \cup J = [Y \setminus B^\circ] \cup J$. Then $f[X \setminus int(f^{-1}(B))] \subseteq (Y \setminus B^\circ) \cup J$, and so $X \setminus int(f^{-1}(B)) \subseteq f^{-1}(Y \setminus B^\circ) \cup f^{-1}(J)$. Hence $f^{-1}(B^\circ \setminus J) \subseteq int(f^{-1}(B))$.

($\leftarrow$) If $A \subseteq X$ then there is a $J \in \mathcal{J}$ with $f^{-1}[int(Y \setminus f(A)) \setminus J] \subseteq int[f^{-1}(Y \setminus f(A))]$. Thus $f^{-1}\left[\left(Y \setminus \overline{f(A)}\right) \setminus J\right] \subseteq X \setminus \overline{f^{-1}(f(A))} \subseteq X \setminus \overline{A}$, and so $\overline{A} \subseteq f^{-1}\left[\overline{f(A)} \cup J\right]$ or, equivalently, $f\left(\overline{A}\right) \subseteq \overline{f(A)} \cup J$, and this implies that $f\left(\overline{A}\right) \setminus \overline{f(A)} \in \mathcal{J}$.

Corollary 2.4 (1) The following propositions are equivalents:

(a) The function $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is $\mathcal{C}$-continuous. (b) For each $V \in \beta$, there exists $J \in \mathcal{J}$ such that $f^{-1}(V \setminus J) \subseteq int(f^{-1}(V))$, this is $f[f^{-1}(V) \setminus int(f^{-1}(V))] \in \mathcal{J}$. (c) For each $B \subseteq Y$, there is a $J \in \mathcal{J}$ such that $\overline{f^{-1}(B)} \subseteq f^{-1}\left(\overline{B} \cup J\right)$ or, equivalently, $f\left[\overline{f^{-1}(B)} \setminus f^{-1}\left(\overline{B}\right)\right] \in \mathcal{J}$.

(d) For each closed $F \subseteq Y$, $f\left[\overline{f^{-1}(F)} \setminus f^{-1}(F)\right] \in \mathcal{J}$. (2) If $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is one-one, then the following statements are equivalents: (a) f is $\mathcal{C}$-continuous. (b) For each $B \subseteq Y$, $\overline{f^{-1}(B)} \setminus f^{-1}\left(\overline{B}\right) \in f^{-1}(\mathcal{J})$. (c) For each closed set $B \subseteq Y$, $f^{-1}(B)$ is closed-$f^{-1}(\mathcal{J})$. (d) For each $V \subseteq Y$, $f^{-1}(V^\circ) \setminus int[f^{-1}(V)] \in f^{-1}(\mathcal{J})$. (e) For each $V \in \beta$, $f^{-1}(V)$

is open-$f^{-1}(\mathcal{J})$. (3) If $\mathcal{I}$ is an ideal in X and if $f : X \to Y$ is an one-one function, then $f : (X, \tau) \to (Y, \beta, f(\mathcal{I}))$ is $\mathcal{C}$-continuous if and only if $f^{-1}(V)$ is open-$\mathcal{I}$, for each $V \in \beta$.

Now we present a characterization of the $\mathcal{C}$-continuous functions, in terms of the frontier operator.

Theorem 2.5 The function $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is $\mathcal{C}$-continuous if and only if, for each $B \subseteq Y$, there exists $J \in \mathcal{J}$ such that $Fr\left[f^{-1}(B)\right] \subseteq f^{-1}(Fr(B) \cup J)$ or, equivalently, $f\left[Fr\left(f^{-1}(B)\right) \setminus f^{-1}(Fr(B))\right] \in \mathcal{J}$.

Proof. ($\to$) If $B \subseteq Y$, there is a $\{J_1, J_2\} \subseteq \mathcal{J}$ such that $\overline{f^{-1}(B)} \subseteq f^{-1}\left(\overline{B} \cup J_1\right)$ and $\overline{f^{-1}(Y \setminus B)} \subseteq f^{-1}\left(\overline{Y \setminus B} \cup J_2\right)$.

So $Fr\left(f^{-1}(B)\right) = \overline{f^{-1}(B)} \cap \overline{X \setminus f^{-1}(B)} \subseteq f^{-1}\left[\left(\overline{B} \cup J_1\right) \cap \left(\overline{Y \setminus B} \cup J_2\right)\right]$
$= f^{-1}\left[Fr(B) \cup \left(\overline{B} \cap J_2\right) \cup \left(J_1 \cap \overline{Y \setminus B}\right) \cup (J_1 \cap J_2)\right] = f^{-1}[Fr(B) \cup J]$,
where $J = \left(\overline{B} \cap J_2\right) \cup \left(J_1 \cap \overline{Y \setminus B}\right) \cup (J_1 \cap J_2) \in \mathcal{J}$.

($\leftarrow$) Suppose that $F \subseteq Y$ is closed. There is a $J \in \mathcal{J}$ such that $Fr\left[f^{-1}(F)\right] \subseteq f^{-1}(Fr(F) \cup J) \subseteq f^{-1}(F \cup J) = f^{-1}(F) \cup f^{-1}(J)$.

Hence
$\overline{f^{-1}(F)} \setminus int\left(f^{-1}(F)\right) \subseteq f^{-1}(F) \cup f^{-1}(J)$, and so $\overline{f^{-1}(F)} \subseteq int\left(f^{-1}(F)\right) \cup f^{-1}(F) \cup f^{-1}(J) = f^{-1}(F) \cup f^{-1}(J)$. Corollary 2.4 implies that f is $\mathcal{C}$-continuous.

The following characterization of $\mathcal{C}$-continuous (and one-one) functions is expressed completely in terms of closed-$\mathcal{I}$ sets.

Theorem 2.6 If $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is one-one, then f is $\mathcal{C}$-continuous if and only if, for each closed-$\mathcal{J}$ set $B \subseteq Y$, it is true that $f^{-1}(B)$ is closed-$f^{-1}(\mathcal{J})$.

Proof. ($\to$) If $B \subseteq Y$ is closed-$\mathcal{J}$, we have that $\overline{B} \setminus B \in \mathcal{J}$, and so $f^{-1}\left(\overline{B}\right) \setminus f^{-1}(B) = f^{-1}\left(\overline{B} \setminus B\right) \in f^{-1}(\mathcal{J})$. In addition, $\overline{f^{-1}(B)} \setminus f^{-1}\left(\overline{B}\right) \in f^{-1}(\mathcal{J})$, by Corollary 2.4. This implies that $\overline{f^{-1}(B)} \setminus f^{-1}(B) \in f^{-1}(\mathcal{J})$.
($\leftarrow$) It is a consequence of Corollary 2.4, given that if $F \subseteq Y$ is closed then F is closed-$\mathcal{J}$.

Corollary 2.7 If $\mathcal{I}$ is an ideal in X and if $f : X \to Y$ is an one-one function, then $f : (X, \tau) \to (Y, \beta, f(\mathcal{I}))$ is $\mathcal{C}$-continuous if and only if $f^{-1}(V)$ is open-$\mathcal{I}$, for each open-$f(\mathcal{I})$ set $V \subseteq Y$.

Theorem 2.8 Suppose that $\mathcal{I}$ is an ideal in X and $f : X \to Y$ is an one-one function. If $\mathcal{J} = \left\{B \subseteq Y : f^{-1}(B) \in \mathcal{I}\right\}$ then $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is $\mathcal{C}$ -continuous if and only if $f^{-1}(B)$ is closed- $\mathcal{I}$, for each closed- $\mathcal{J}$ set $B \subseteq Y$.

Proof. $(\to)$ If $B \subseteq Y$ is closed- $\mathcal{J}$ then $\overline{B} \backslash B \in \mathcal{J}$, and so $f^{-1}\left(\overline{B}\right) \backslash f^{-1}(B) \in$ $\mathcal{I}$. Since f is $\mathcal{C}$-continuous we have that $\overline{f^{-1}(B)} \backslash f^{-1}\left(\overline{B}\right) \in f^{-1}(\mathcal{J}) \subseteq \mathcal{I}$. Given that $\overline{f^{-1}(B)} \backslash f^{-1}(B) \subseteq \left[f^{-1}\left(\overline{B}\right) \backslash f^{-1}(B)\right] \cup \left[\overline{f^{-1}(B)} \backslash f^{-1}\left(\overline{B}\right)\right]$, then $\overline{f^{-1}(B)} \backslash f^{-1}(B) \in \mathcal{I}$.
$(\leftarrow)$ If $B \subseteq Y$ then $\overline{f^{-1}\left(\overline{B}\right)} \backslash f^{-1}\left(\overline{B}\right) \in \mathcal{I}$, since $\overline{B}$ is closed- $\mathcal{J}$, and so $\overline{f^{-1}(B)} \backslash f^{-1}\left(\overline{B}\right) \in \mathcal{I}$. There exists $I \in \mathcal{I}$ with $\overline{f^{-1}(B)} \backslash f^{-1}\left(\overline{B}\right) = I = f^{-1}(f(I))$. Moreover $f(I) \in \mathcal{J}$.

The following two properties of the $\mathcal{C}$-continuity are related with restrictions of functions.

Theorem 2.9 If $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is $\mathcal{C}$ -continuous and $A \subseteq X$, then the function $f_A : (A, \tau_A) \to (Y, \beta, \mathcal{J})$ is $\mathcal{C}$ -continuous, where f_A is the restriction of f to A .

Proof. If $V \in \beta$, there is a $J \in \mathcal{J}$ such that $f^{-1}(V) \subseteq int\left(f^{-1}(V)\right) \cup f^{-1}(J)$. Thus $f_A^{-1}(V) = A \cap f^{-1}(V) \subseteq \left[A \cap int\left(f^{-1}(V)\right)\right] \cup \left[A \cap f^{-1}(J)\right] = \left[A \cap int\left(f^{-1}(V)\right)\right] \cup f_A^{-1}(J)$. But $A \cap int\left(f^{-1}(V)\right) \subseteq int_{\tau_A}\left[A \cap f^{-1}(V)\right] = int_{\tau_A}\left(f_A^{-1}(V)\right)$. Hence $f_A^{-1}(V) \subseteq int_{\tau_A}\left(f_A^{-1}(V)\right) \cup f_A^{-1}(J)$.

Theorem 2.10 Suppose that (X, τ) is a topological space and that $X = U \cup V$, where $\{U, V\} \subseteq \tau$. If $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is a function such that $f_U : (U, \tau_U) \to (Y, \beta, \mathcal{J})$ and $f_V : (V, \tau_V) \to (Y, \beta, \mathcal{J})$ are $\mathcal{C}$ -continuous, then f is $\mathcal{C}$ -continuous.

Proof. If $W \in \beta$, there is a $\{J_1, J_2\} \subseteq \mathcal{J}$ such that $f_U^{-1}(W) \subseteq int_{\tau_U}\left(f_U^{-1}(W)\right) \cup f_U^{-1}(J_1)$ and $f_V^{-1}(W) \subseteq int_{\tau_V}\left(f_V^{-1}(W)\right) \cup f_V^{-1}(J_2)$. Then $f^{-1}(W) = f_U^{-1}(W) \cup f_V^{-1}(W) \subseteq int_{\tau_U}\left(f_U^{-1}(W)\right) \cup int_{\tau_V}\left(f_V^{-1}(W)\right) \cup f_U^{-1}(J_1) \cup f_V^{-1}(J_2)$. Given that U and V are open, $int_{\tau_U}\left(f_U^{-1}(W)\right) = U \cap int\left(f^{-1}(W)\right)$ and $int_{\tau_V}\left(f_V^{-1}(W)\right) = V \cap int\left(f^{-1}(W)\right)$. Thus $f^{-1}(W) \subseteq \left[U \cap int\left(f^{-1}(W)\right)\right]$ $\cup \left[V \cap int\left(f^{-1}(W)\right)\right] \cup f^{-1}(J_1 \cup J_2) = int\left(f^{-1}(W)\right) \cup f^{-1}(J_1 \cup J_2)$.

Recall that if $A \subseteq X$, the characteristic function, associated with A, is the function $\chi_A : X \to \{0, 1\}$ defined by: $\chi_A(x) = 1$, if $x \in A$, and $\chi_A(x) = 0$, if $x \notin A$.

Theorem 2.11 If (X, τ) is a topological space, $A \subseteq X$, $\mathcal{J}_1 = \{\emptyset, \{0\}\}$ and $\mathcal{J}_2 = \{\emptyset, \{1\}\}$ then:
(1) The function $\chi_A : (X, \tau) \to (\{0, 1\}, \mathcal{P}(\{0, 1\}), \mathcal{J}_1)$ is $\mathcal{C}$-continuous if and only if A is open. (2) The function $\chi_A : (X, \tau) \to (\{0, 1\}, \mathcal{P}(\{0, 1\}), \mathcal{J}_2)$ is $\mathcal{C}$-continuous if and only if A is closed.

Proof. (1) ($\to$) There is a $J \in \mathcal{J}_1$ such that $A = \chi_A^{-1}(\{1\}) \subseteq \chi_A^{-1}(J) \cup int\left(\chi_A^{-1}(\{1\})\right) = A^\circ \cup \chi_A^{-1}(J) \subseteq A^\circ \cup (X \backslash A)$. Hence $A \subseteq A^\circ$, and then A is open. ($\leftarrow$) It is sufficient to note that: (a) $\chi_A^{-1}(\{0\}) \subseteq \chi_A^{-1}(\{0\}) \cup int\left(\chi_A^{-1}(\{0\})\right)$, and (b) $\chi_A^{-1}(\{1\}) = A \subseteq \chi_A^{-1}(J) \cup A = \chi_A^{-1}(J) \cup int\left(\chi_A^{-1}(\{1\})\right)$, for all $J \in \mathcal{J}_1$.
  (2) It is similar to (1).

Theorem 2.12 Let (Y, β) be a topological space, $\mathcal{B}$ a base for β and $\mathcal{J}$ an ideal in Y. The function $f : (X, \tau) \to (Y, \beta, \mathcal{J}^\otimes)$ is $\mathcal{C}$-continuous if and only if, for each $B \in \mathcal{B}$, there exists $J \in \mathcal{J}^\otimes$ such that $f^{-1}(B) \subseteq int\left(f^{-1}(B)\right) \cup f^{-1}(J)$ or, equivalently, $f\left[f^{-1}(B) \backslash int\left(f^{-1}(B)\right)\right] \in \mathcal{J}^\otimes$.

Proof. ($\to$) It is clear.
  ($\leftarrow$) If $V \in \beta$ then there exists $\{B_\alpha : \alpha \in \Delta\} \subseteq \mathcal{B}$, such that $V = \bigcup_{\alpha \in \Delta} B_\alpha$. If $\alpha \in \Delta$, there is a $J_\alpha \in \mathcal{J}^\otimes$ with $f^{-1}(B_\alpha) \subseteq int\left(f^{-1}(B_\alpha)\right) \cup f^{-1}(J_\alpha)$. If $J = \bigcup_{\alpha \in \Delta} J_\alpha$ then $J \in \mathcal{J}^\otimes$.
Now, $f^{-1}(V) = \bigcup_{\alpha \in \Delta} f^{-1}(B_\alpha) \subseteq \bigcup_{\alpha \in \Delta} \left[int\left(f^{-1}(B_\alpha)\right) \cup f^{-1}(J_\alpha)\right] \subseteq int\left(f^{-1}(V)\right) \cup f^{-1}(J)$.

Theorem 2.13 Let (Y, β) be a topological space, $\mathcal{S}$ a sub-base for β and $\mathcal{J}$ an ideal in Y. The function $f : (X, \tau) \to (Y, \beta, \mathcal{J}^\otimes)$ is $\mathcal{C}$-continuous if and only if for each $S \in \mathcal{S}$ there exists $J \in \mathcal{J}^\otimes$ such that $f^{-1}(S) \subseteq int\left(f^{-1}(S)\right) \cup f^{-1}(J)$.

Proof. ($\to$) It is clear.
  ($\leftarrow$) By Theorem 2.12, it is enough to see that if $\{S_1, S_2\} \subseteq \mathcal{S}$, there exists $J \in \mathcal{J}^\otimes$ such that $f^{-1}(S_1 \cap S_2) \subseteq int\left(f^{-1}(S_1 \cap S_2)\right) \cup f^{-1}(J)$. If

$\{S_1, S_2\} \subseteq \mathcal{S}$, there is a $\{J_1, J_2\} \subseteq \mathcal{J}^{\otimes}$ such that $f^{-1}(S_i) \subseteq int(f^{-1}(S_i)) \cup f^{-1}(J_i)$, for all $i \in \{1, 2\}$.

Thus $f^{-1}(S_1 \cap S_2) = f^{-1}(S_1) \cap f^{-1}(S_2) \subseteq [int(f^{-1}(S_1)) \cup f^{-1}(J_1)] \cap [int(f^{-1}(S_2)) \cup f^{-1}(J_2)] = [int(f^{-1}(S_1)) \cap int(f^{-1}(S_2))] \cup A \cup B \cup f^{-1}(J_3)$, where $J_3 = J_1 \cap J_2$, $A = int(f^{-1}(S_1)) \cap f^{-1}(J_2)$ and $B = f^{-1}(J_1) \cap int(f^{-1}(S_2))$.

Hence $f^{-1}(S_1 \cap S_2) \subseteq int[f^{-1}(S_1 \cap S_2)] \cup f^{-1}(J_2 \cup J_1)$, and $J_2 \cup J_1 \in \mathcal{J}^{\otimes}$.

Theorem 2.14 If $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ and $g : (Y, \beta) \to (Z, \gamma, \mathcal{L})$ are $\mathcal{C}$-continuous, and if $g(\mathcal{J}) \subseteq \mathcal{L}$ then $g \circ f$ is $\mathcal{C}$-continuous.

Proof. If $A \subseteq X$, there are $J \in \mathcal{J}$ and $L \in \mathcal{L}$ such that $f(\overline{A}) \subseteq \overline{f(A)} \cup J$ and $g(\overline{f(A)}) \subseteq \overline{(g \circ f)(A)} \cup L$. Thus $(g \circ f)(\overline{A}) = g(f(\overline{A})) \subseteq g(\overline{f(A)}) \cup g(J) \subseteq \overline{(g \circ f)(A)} \cup L \cup g(J)$, with $L \cup g(J) \in \mathcal{L}$.

Corollary 2.15 If $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ and $g : (Y, \beta) \to (Z, \gamma, g(\mathcal{J}))$ are $\mathcal{C}$-continuous, then $g \circ f$ is $\mathcal{C}$-continuous.

The example that follows shows us that, in general, the composition of $\mathcal{C}$-continuous functions is not $\mathcal{C}$-continuous.

Example 2.16 If $X = Y = Z = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$, $\beta = \{\emptyset, Y, \{b\}, \{a, b\}, \{b, c\}\}$, $\gamma = \{\emptyset, Z, \{c\}, \{a, c\}, \{b, c\}\}$, $\mathcal{L} = \{\emptyset, \{a\}\}$, $\mathcal{J} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, and if $f : X \to Y$ and $g : Y \to Z$ are defined by $f(a) = c$, $f(b) = a$, $f(c) = b$, $g(a) = a = g(c)$ and $g(b) = b$, it is easy to see that f and g are $\mathcal{C}$-continuous. However, since $(g \circ f)(\overline{\{a\}}) \setminus \overline{(g \circ f)(\{a\})} = \{a, b\} \setminus \{a\} = \{b\} \notin \mathcal{L}$, we have that $g \circ f$ is not $\mathcal{C}$-continuous.

Theorem 2.17 If $f : (X, \tau) \to (Y, \beta)$ is a function, $\{X_\alpha\}_{\alpha \in \Lambda}$ is a locally finite collection of closed subsets of X whose union is X, and if $\mathcal{J}$ is an ideal on Y, then $f : (X, \tau) \to (Y, \beta, \mathcal{J}^{\otimes})$ is $\mathcal{C}$-continuous if and only if each restriction $f_{X_\alpha} : (X_\alpha, \tau_\alpha) \to (Y, \beta, \mathcal{J}^{\otimes})$ is $\mathcal{C}$-continuous. Here $\tau_\alpha = \tau_{X_\alpha}$.

Proof. ($\leftarrow$) For each $\alpha \in \Lambda$, we will denote the function f_{X_α} simply by f_α. Suppose that $F \subseteq Y$ is closed. Since $f^{-1}(F) = \bigcup_{\alpha \in \Lambda} f_\alpha^{-1}(F)$ and $\{f_\alpha^{-1}(F)\}_{\alpha \in \Lambda}$ is locally finite, we have that $\overline{f^{-1}(F)} = \overline{\bigcup_{\alpha \in \Lambda} f_\alpha^{-1}(F)} =$

$\bigcup_{\alpha \in \Lambda} \overline{f_\alpha^{-1}(F)}$. Now, for each $\alpha \in \Lambda$, there is a $J_\alpha \in \mathcal{J}^\otimes$ such that $adh_{\tau_\alpha}\left(f_\alpha^{-1}(F)\right) \subseteq$ $f_\alpha^{-1}(F \cup J_\alpha)$. Given that each X_α is closed, $adh_{\tau_\alpha}\left(f_\alpha^{-1}(F)\right) = \overline{f_\alpha^{-1}(F)}$.

Hence $\overline{f^{-1}(F)} \subseteq \bigcup_{\alpha \in \Lambda} f_\alpha^{-1}(F) \cup f^{-1}\left(\bigcup_{\alpha \in \Lambda} J_\alpha\right) = f^{-1}(F) \cup f^{-1}\left(\bigcup_{\alpha \in \Lambda} J_\alpha\right) =$ $f^{-1}\left(F \cup \bigcup_{\alpha \in \Lambda} J_\alpha\right)$, with $\bigcup_{\alpha \in \Lambda} J_\alpha \in \mathcal{J}^\otimes$.

$(\rightarrow)$ It is a consequence of Theorem 2.9.

Notation 2.18 If $\mathcal{I}$ is an ideal in X and $\mathcal{J}$ is an ideal in Y, then $\mathcal{I} \otimes \mathcal{J}$ [10] is the set of all $D \subseteq X \times Y$ such that there exist $I \in \mathcal{I}$, $A \subseteq X$, $J \in \mathcal{J}$ and $B \subseteq Y$, with $D \subseteq (A \times J) \cup (I \times B)$.

It is shown in [10] that $\mathcal{I} \otimes \mathcal{J}$ is an ideal in $X \times Y$, and that if A is open-$\mathcal{I}$ in $(X, \tau, \mathcal{I})$ and B is open-$\mathcal{J}$ in $(Y, \beta, \mathcal{J})$ then $A \times B$ is open-$(\mathcal{I} \otimes \mathcal{J})$.

It is possible to build new $\mathcal{C}$-continuous functions from some previously known ones, as we will show in the two theorems that follow.

Theorem 2.19 If $f : (X, \tau) \rightarrow (Y, \beta, \mathcal{J})$ and $g : (X, \tau) \rightarrow (Z, \gamma, \mathcal{L})$ are $\mathcal{C}$-continuous, then $h : (X, \tau) \rightarrow \left(Y \times Z, \beta \times \gamma, (\mathcal{J} \otimes \mathcal{L})^\otimes\right)$, defined by $h(x) = (f(x), g(x))$ for all $x \in X$, is $\mathcal{C}$-continuous.

Proof. If $U \in \beta$ and $V \in \gamma$ then there are $J \in \mathcal{J}$ and $L \in \mathcal{L}$, such that $f^{-1}(U) \subseteq int\left(f^{-1}(U)\right) \cup f^{-1}(J)$ and $g^{-1}(V) \subseteq int\left(g^{-1}(V)\right) \cup g^{-1}(L)$.

Now, $h^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V) \subseteq \left[int\left(f^{-1}(U)\right) \cup f^{-1}(J)\right] \cap \left[int\left(g^{-1}(V)\right) \cup g^{-1}(L)\right] = int\left[f^{-1}(U) \cap g^{-1}(V)\right] \cup \left[int\left(f^{-1}(U)\right) \cap g^{-1}(L)\right] \cup \left[f^{-1}(J) \cap int\left(g^{-1}(V)\right)\right] \cup \left[f^{-1}(J) \cap g^{-1}(L)\right] \subseteq int\left[h^{-1}(U \times V)\right] \cup h^{-1}(J \times L) \cup h^{-1}(U \times L) \cup h^{-1}(J \times V) = int\left[h^{-1}(U \times V)\right] \cup h^{-1}\left[(J \times L) \cup (U \times L) \cup (J \times V)\right]$. Moreover $J \times L \in \mathcal{J} \otimes \mathcal{L}$ and $(U \times L) \cup (J \times V) \in \mathcal{J} \otimes \mathcal{L} \subseteq (\mathcal{J} \otimes \mathcal{L})^\otimes$. Theorem 2.12 implies that h is $\mathcal{C}$-continuous.

Theorem 2.20 If $f : (X, \tau) \rightarrow (Y, \beta, \mathcal{J})$ is $\mathcal{C}$-continuous and if $\mathcal{J}_\tau = \{W \subseteq X \times Y : \text{ there are } U \in \tau \text{ and } J \in \mathcal{J} \text{ such that } W \subseteq U \times J\}$, then $\mathcal{J}_\tau$ is an ideal in $X \times Y$ and the function $g : (X, \tau) \rightarrow \left(X \times Y, \tau \times \beta, (\mathcal{J}_\tau)^\otimes\right)$, defined by $g(x) = (x, f(x))$, is $\mathcal{C}$-continuous.

Proof. It is easy to be established that $\mathcal{J}_\tau$ is an ideal in $X \times Y$. Suppose that $U \in \tau$ and $V \in \beta$. There is a $J \in \mathcal{J}$ such that $f^{-1}(V) \subseteq int\left(f^{-1}(V)\right) \cup f^{-1}(J)$. Now, $g^{-1}(U \times V) = U \cap f^{-1}(V) \subseteq U \cap \left[int\left(f^{-1}(V)\right) \cup f^{-1}(J)\right] = \left[U \cap int\left(f^{-1}(V)\right)\right] \cup \left[U \cap f^{-1}(J)\right] = int\left(U \cap f^{-1}(V)\right) \cup g^{-1}(U \times J) =$

$int\left(g^{-1}\left(U\times V\right)\right)\cup g^{-1}\left(U\times J\right)$, and $U\times J\in\mathcal{J}_{\tau}\subseteq\left(\mathcal{J}_{\tau}\right)^{\otimes}$. Theorem 2.12 implies that g is $\mathcal{C}$-continuous.

Recall that, if $\mathcal{I}$ and $\mathcal{J}$ are ideals in X, then the ideal $\mathcal{I}\vee\mathcal{J}$ is the set $\{I\cup J : I\in\mathcal{I}$ and $J\in\mathcal{J}\}$.

Lemma 2.21 If $\mathcal{I}$ and $\mathcal{J}$ are ideals in X , $A\subseteq X$ is open- $\mathcal{I}$ in $(X,\tau,\mathcal{I})$ and $B\subseteq X$ is open- $\mathcal{J}$ in $(X,\tau,\mathcal{J})$, then $A\cap B$ is open- $(\mathcal{I}\vee\mathcal{J})$.

Proof. Since $A\backslash A^{\circ}\in\mathcal{I}$ and $B\backslash B^{\circ}\in\mathcal{J}$, we have that $(A\cap B)\backslash int\left(A\cap B\right)\subseteq(A\backslash A^{\circ})\cup(B\backslash B^{\circ})\in\mathcal{I}\vee\mathcal{J}$, and so $(A\cap B)\backslash int\left(A\cap B\right)\in\mathcal{I}\vee\mathcal{J}$.

If $(X,\tau,\mathcal{I})$ is an ideal space, the set $\tau\cup\mathcal{I}$ is a base for a topology $\tau\oplus\mathcal{I}$ in X. It is observed that $\tau\oplus\mathcal{I}=\{V\cup I : V\in\tau$ and $I\in\mathcal{I}^{\otimes}\}=\tau\oplus\mathcal{I}^{\otimes}$ [10]. If $\mathcal{J}$ is an ideal in Y and if $f : X\rightarrow Y$ is a function, we denote the set $\left\{A\subseteq X : A\subseteq f^{-1}\left(J\right)$ for some $J\in\mathcal{J}\right\}$ by $\mathcal{I}_{f,\mathcal{J}}$. It is noted that $\mathcal{I}_{f,\mathcal{J}}$ is an ideal in X and that $\{f^{-1}\left(J\right) : J\in\mathcal{J}\}\subseteq\mathcal{I}_{f,\mathcal{J}}$.

The following theorem shows some properties of the $\mathcal{C}$-continuous functions, in the case where the codomain is a Hausdorff space.

Theorem 2.22 (1) If $f : (X,\tau)\rightarrow(Y,\beta,\mathcal{J})$ and $g : (X,\tau)\rightarrow(Y,\beta,\mathcal{J})$ are $\mathcal{C}$-continuous functions and if (Y,β) is T_2, the set $A=\{x\in X : f\left(x\right)=g\left(x\right)\}$ is closed in the space $(X,\tau\oplus\left(\mathcal{I}_{f,\mathcal{J}}\vee\mathcal{I}_{g,\mathcal{J}}\right))$.
(2) If $f : (X,\tau)\rightarrow(Y,\beta,\mathcal{J})$ is a $\mathcal{C}$-continuous function and if (Y,β) is T_2, the set $A=\{(u,v)\in X\times X : f\left(u\right)=f\left(v\right)\}$ is closed in the space $(X\times X,(\tau\times\tau)\oplus\left(\mathcal{I}_{f,\mathcal{J}}\otimes\mathcal{I}_{f,\mathcal{J}}\right))$.

Proof. (1) Suppose that $u\in X\backslash A$. There is a $\{U,V\}\subseteq\tau$ such that $f\left(u\right)\in U$, $g\left(u\right)\in V$ and $U\cap V=\emptyset$. Then $u\in f^{-1}\left(U\right)\cap g^{-1}\left(V\right)$, $f^{-1}\left(U\right)\cap g^{-1}\left(V\right)\cap A=\emptyset$ and, by hypothesis, Lemma 2.21 and Corollary 2.4, $f^{-1}\left(U\right)\cap g^{-1}\left(V\right)$ is open-$(\mathcal{I}_{f,\mathcal{J}}\vee\mathcal{I}_{g,\mathcal{J}})$. Thus $f^{-1}\left(U\right)\cap g^{-1}\left(V\right)\in\tau\oplus\left(\mathcal{I}_{f,\mathcal{J}}\vee\mathcal{I}_{g,\mathcal{J}}\right)$.
(2) Suppose that $(u,v)\in(X\times X)\backslash A$. There exists $\{U,V\}\subseteq\beta$ such that $f\left(u\right)\in U$, $f(v)\in V$ and $U\cap V=\emptyset$. Thus $(u,v)\in f^{-1}\left(U\right)\times f^{-1}\left(V\right)$, $A\cap\left[f^{-1}\left(U\right)\times f^{-1}\left(V\right)\right]=\emptyset$ and $f^{-1}\left(U\right)\times f^{-1}\left(V\right)$ is open-$(\mathcal{I}_{f,\mathcal{J}}\otimes\mathcal{I}_{f,\mathcal{J}})$, and so $f^{-1}\left(U\right)\times f^{-1}\left(V\right)\in(\tau\times\tau)\oplus\left(\mathcal{I}_{f,\mathcal{J}}\otimes\mathcal{I}_{f,\mathcal{J}}\right)$.

In our last property for this new type of weak continuity, we present the smallest ideal $\mathcal{J}$ in Y for which a given function $f : (X,\tau)\rightarrow(Y,\beta)$ is $\mathcal{C}$-continuous.

Theorem 2.23 If $f : (X, \tau) \to (Y, \beta)$ is a function and if

$$\mathcal{J}_f = \left\{ B \subseteq Y : \text{there is a finite } \Delta_B \subseteq \mathcal{P}(X) \text{ with } B \subseteq \bigcup_{A \in \Delta_B} \left[f\left(\overline{A}\right) \setminus \overline{f(A)} \right] \right\}$$

then $\mathcal{J}_f$ is the smallest ideal $\mathcal{J}$ in Y such that $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is $\mathcal{C}$-continuous.

Proof. It is evident that $\mathcal{J}_f$ is an ideal in Y. Given that $f\left(\overline{A}\right) \setminus \overline{f(A)} \in \mathcal{J}_f$, for each $A \subseteq X$, then $f : (X, \tau) \to (Y, \beta, \mathcal{J}_f)$ is a $\mathcal{C}$-continuous function. Now, suppose that $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is $\mathcal{C}$-continuous. If $B \in \mathcal{J}_f$ then there is a finite $\Delta_B \subseteq \mathcal{P}(X)$ with $B \subseteq \bigcup_{A \in \Delta_B} \left[f\left(\overline{A}\right) \setminus \overline{f(A)} \right]$. Since $f\left(\overline{A}\right) \setminus \overline{f(A)} \in \mathcal{J}$, for each $A \in \Delta_B$, we can conclude that $B \in \mathcal{J}$. In this way $\mathcal{J}_f \subseteq \mathcal{J}$.

The example that follows shows us that $\mathcal{C}$-continuous and $\mathcal{J}$-continuous are independent concepts.

Example 2.24 Let $X = Y = \{a, b\}$, $\tau = \{\emptyset, X\}$, $\beta = \{\emptyset, \{a\}, Y\}$ and $f : X \to Y$ the function defined by $f(a) = b$ and $f(b) = a$. Then:
(1) If $\mathcal{J} = \{\emptyset, \{a\}\}$, it is not a problem to prove that f is $\mathcal{C}$-continuous. However $f(b) = a \in \{a\}$, $\{a\} \in \beta$, but $f(X) \setminus \{a\} = \{b\} \notin \mathcal{J}$. Hence f is not $\mathcal{J}$-continuous.
(2) If $\mathcal{J} = \{\emptyset, \{b\}\}$, it is easy to see that f is $\mathcal{J}$-continuous. Now, since $f\left(\overline{\{a\}}\right) \setminus \overline{f(\{a\})} = f(X) \setminus \overline{\{b\}} = \{a\} \notin \mathcal{J}$, we conclude that f is not $\mathcal{C}$-continuous.

Next we show that $\mathcal{C}$-continuous and $\mathcal{J}_w$-continuous are independent concepts.

Example 2.25 (1) Suppose that $X = Y = \{a, b, c\}$, $\beta = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$, $\tau = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$, $\mathcal{J} = \{\emptyset, \{a\}\}$ and that $f : X \to Y$ is defined by: $f(a) = a$, $f(b) = b = f(c)$. It is easy to see that f is $\mathcal{C}$-continuous. Given that $f(a) = a \in \{a\}$, $\{a\} \in \beta$, $f(\{a, b\}) \setminus \overline{\{a\}} = \{b\} \notin \mathcal{J}$ and $f(X) \setminus \overline{\{a\}} = \{b\}$, then f is not $\mathcal{J}_w$-continuous.
(2) Let X, Y, τ, β and $\mathcal{J}$ be as in part (1). If $f : X \to Y$ is defined by: $f(a) = b = f(c)$, and $f(b) = c$, it is not a problem to prove that f is $\mathcal{J}_w$-continuous. However, given that $f\left(\overline{\{b\}}\right) \setminus \overline{f(\{b\})} = \{b, c\} \setminus \overline{\{c\}} = \{b\} \notin \mathcal{J}$, we have that f is not $\mathcal{C}$-continuous.

3. $\mathcal{C}$-closed and $\mathcal{C}$-open functions

Through the $\mathcal{I}$-open sets, Abd El Monsef et al. defines the $\mathcal{J}$-open functions. The authors consider the case in which an ideal is defined in the codomain of the function. Under the same requirement, in this section we define the $\mathcal{C}$-open functions, and we show that this new concept is independent of the concept introduced by them. Recall that if $\mathcal{J}$ is an ideal in Y, a function $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is said to be $\mathcal{J}$ -open (resp. $\mathcal{J}$ -closed) [1] if, for each $U \in \tau$ (resp. U is closed), we have that $f(U)$ is $\mathcal{J}$-open (resp. $\mathcal{J}$-closed).

Definition 3.1 If $\mathcal{J}$ is an ideal in Y, a function $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is said to be:
(1) $\mathcal{C}$ -closed if, for each $A \subseteq X$, we have that $\overline{f(A)} \backslash f\left(\overline{A}\right) \in \mathcal{J}$.
(2) $\mathcal{C}$ -open if, for each $A \subseteq X$, we have that $f(A^\circ) \backslash int(f(A)) \in \mathcal{J}$.
    It is clear that if f is closed (or open) then f is $\mathcal{C}$-closed (or $\mathcal{C}$-open) and that $f : (X, \tau) \to (Y, \beta)$ is closed (or open) if and only if $f : (X, \tau) \to (Y, \beta, \{\emptyset\})$ is $\mathcal{C}$-closed (or $\mathcal{C}$-open). It is immediate to see that $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is: (1) $\mathcal{C}$-open if and only if, for each $U \in \tau$, $f(U)$ is open-$\mathcal{J}$, and (2) $\mathcal{C}$-closed if and only if, for all closed set $F \subseteq X$, $f(F)$ is closed-$\mathcal{J}$. Moreover, if f is $\mathcal{C}$-open then $f : (X, \tau) \to (Y, \beta \oplus \mathcal{J})$ is open, and $f : (X, \tau) \to (Y, \beta, \mathcal{J}^{\otimes})$ is $\mathcal{C}$-open if and only if $f : (X, \tau) \to (Y, \beta \oplus \mathcal{J})$ is open.

Examples 3.2
(1) Each function $f : (X, \tau) \to (Y, \beta, \mathcal{J} = \mathcal{P}(Y))$ is $\mathcal{C}$-closed and $\mathcal{C}$-open.
(2) If $\beta = \{\emptyset, \mathbf{R}, \{0\}\}$ and $\mathcal{J} = \mathcal{P}(\{-1, 0, 1\})$ then the function $f : (\mathbf{R}, \mathcal{U}) \to (\mathbf{R}, \beta, \mathcal{J})$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0, \end{cases}$$

is $\mathcal{C}$-open, but f is not open. Moreover f is not $\mathcal{C}$-closed, because $\overline{f(\mathbf{R})} \backslash f\left(\overline{\mathbf{R}}\right)$ $= \mathbf{R} \backslash \{-1, 0, 1\} \notin \mathcal{J}$.
(3) If $\beta = \{\emptyset, \mathbf{R}\} \cup \{(r, \infty) : r \in \mathbf{R}\}$ and $\mathcal{J} = \mathcal{P}((-\infty, 1])$, then the function $f : (\mathbf{R}, \mathcal{U}) \to (\mathbf{R}, \beta, \mathcal{J})$, defined as in example (2), is not closed, but f is $\mathcal{C}$-closed since if $F \subseteq \mathbf{R}$ then $\overline{f(F)} \subseteq (-\infty, 1]$, and so $\overline{f(F)} \backslash f\left(\overline{F}\right) \in \mathcal{J}$.
(4) Consider the function $f : (\mathbf{R}, \mathcal{U}) \to (\mathbf{R}, \mathcal{U}, \mathcal{J} = \mathcal{P}(\{0\}))$ defined by $f(x) = [x]$, where $[x]$ denotes the integer part of x.

Since $f\left(int\left([1,2]\right)\right)\setminus int\left(f\left([1,2]\right)\right) = \{1\} \notin \mathcal{J}$, we conclude that f is not $\mathcal{C}$-open.

Since that open $\rightarrow \mathcal{C}$-open, and that $\mathcal{J}$-open and open are independent concepts [1], we have that $\mathcal{C}$-open $\not\rightarrow \mathcal{J}$-open. The next example shows that $\mathcal{J}$-open $\not\rightarrow \mathcal{C}$-open.

Example 3.3 Suppose that $X = Y = \{a, b, c, d\}$ and that $f : X \rightarrow Y$ is defined by $f(x) = c$, for each $x \in X$. Let $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\beta = \{\emptyset, Y, \{a, d\}, \{b, c\}\}$ and $\mathcal{J} = \{\emptyset, \{b\}, \{d\}, \{b, d\}\}$. Given that $\{a\} \in \tau$ and $f\left(\{a\}\right)\setminus int\left(f\left(\{a\}\right)\right) = \{c\} \notin \mathcal{J}$, we have that f is not $\mathcal{C}$-open. Since, for each $U \in \tau\setminus\{\emptyset\}$, $f\left(U\right) = \{c\} \subseteq \{b, c\} = int\left[(f\left(U\right))^*\right]$, we conclude that f is $\mathcal{J}$-open.

Theorem 3.4 Suppose that (X, τ) and (Y, β) are topological spaces, $\mathcal{B}$ is a base for τ and that $\mathcal{J}$ is an ideal in Y. Then the function $f : (X, \tau) \rightarrow (Y, \beta, \mathcal{J}^{\otimes})$ is $\mathcal{C}$-open if and only $f(B)$ is open-$\mathcal{J}^{\otimes}$, for each $B \in \mathcal{B}$.

Proof. ($\rightarrow$) It is clear.

($\leftarrow$) If $V \in \tau$, there exists $\{V_\alpha : \alpha \in \Lambda\} \subseteq \mathcal{B}$ such that $V = \bigcup_{\alpha \in \Lambda} V_\alpha$. For each $\alpha \in \Lambda$ there exists $J_\alpha \in \mathcal{J}^{\otimes}$ such that $f\left(V_\alpha\right) = int\left(f\left(V_\alpha\right)\right) \cup J_\alpha$. Then $f(V) = \bigcup_{\alpha \in \Lambda} f(V_\alpha) = \bigcup_{\alpha \in \Lambda} int\left(f\left(V_\alpha\right)\right) \cup \bigcup_{\alpha \in \Lambda} J_\alpha \subseteq int(f(V)) \cup \bigcup_{\alpha \in \Lambda} J_\alpha$. Hence $f(V)\setminus int(f(V)) \in \mathcal{J}^{\otimes}$.

Theorem 3.5 (1) Suppose that $f : (X, \tau) \rightarrow (Y, \beta, \mathcal{J})$ and $g : (Y, \beta) \rightarrow (Z, \gamma, \mathcal{L})$ are $\mathcal{C}$-open and that $g(\mathcal{J}) \subseteq \mathcal{L}$. Then $g \circ f$ is $\mathcal{C}$-open.
(2) If $f : (X, \tau) \rightarrow (Y, \beta)$ is open and if $g : (Y, \beta) \rightarrow (Z, \gamma, \mathcal{L})$ is $\mathcal{C}$-open, then $g \circ f$ is $\mathcal{C}$-open.

Proof. (1) If $U \in \tau$ then there is a $J \in \mathcal{J}$ such that $f(U)\setminus int\left(f(U)\right) = J$. Then $f(U) = int\left(f(U)\right) \cup J$. There exists $L \in \mathcal{L}$ with $g\left(int\left(f(U)\right)\right) = int\left(g\left(int\left(f(U)\right)\right)\right) \cup L$.

Hence $g\left(f\left(U\right)\right) = g\left(int\left(f(U)\right)\right) \cup g\left(J\right) = int\left(g\left(int\left(f(U)\right)\right)\right) \cup L \cup g\left(J\right) \subseteq int\left(g\left(f(U)\right)\right) \cup L \cup g\left(J\right)$, and $L \cup g\left(J\right) \in \mathcal{L}$.

This implies that $(g \circ f)\left(U\right)\setminus int\left((g \circ f)\left(U\right)\right) \in \mathcal{L}$.

(2) It is a consequence of (1).

The following example shows that the composition of $\mathcal{C}$-open functions may not be a $\mathcal{C}$-open function.

Example 3.6 Suppose that $X = Y = Z = \{a, b, c, d\}$ and that $f : X \to Y$ and $g : Y \to Z$ are functions defined by $f(a) = b = f(b)$, $f(c) = c = f(d)$ and $g(a) = a$, $g(b) = c$, $g(c) = b$, $g(d) = d$.

If $\beta = \{\emptyset, Y, \{b\}, \{a, d\}, \{a, b, d\}\}$, $\tau = \{\emptyset, X, \{a, b, c\}, \{a, c, d\}, \{a, c\}\}$, $\gamma = \{\emptyset, Z, \{c\}\}$, $\mathcal{J} = \{\emptyset, \{c\}\}$, $\mathcal{L} = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$, we have that $g \circ f$ is not $\mathcal{C}$-open, because $(g \circ f)(\{a, b, c\}) \setminus int\,[(g \circ f)(\{a, b, c\})] = \{b\} \notin \mathcal{L}$. However f and g are $\mathcal{C}$-open.

Theorem 3.7 If $\mathcal{I}$ is an ideal in X then the function $f : (X, \tau) \to (Y, \beta, f(\mathcal{I}))$ is $\mathcal{C}$-closed if and only if, for each closed-$\mathcal{I}$ set $A \subseteq X$, it is true that $f(A)$ is closed-$f(\mathcal{I})$.

Proof. ($\to$) Suppose that $A \subseteq X$ is closed-$\mathcal{I}$. Since $\overline{A} \setminus A \in \mathcal{I}$ and $f\left(\overline{A}\right) \setminus f(A) \subseteq f\left(\overline{A} \setminus A\right) \in f(\mathcal{I})$, then $f\left(\overline{A}\right) \setminus f(A) \in f(\mathcal{I})$. Now, given that f is $\mathcal{C}$-closed, we have that $\overline{f\left(\overline{A}\right)} \setminus f\left(\overline{A}\right) \in f(\mathcal{I})$. Moreover, $\overline{f(A)} \setminus f(A) \subseteq \overline{f\left(\overline{A}\right)} \setminus f(A) \subseteq \left[\overline{f\left(\overline{A}\right)} \setminus f\left(\overline{A}\right)\right] \cup \left[f\left(\overline{A}\right) \setminus f(A)\right] \in f(\mathcal{I})$, and so $\overline{f(A)} \setminus f(A) \in f(\mathcal{I})$.

($\leftarrow$) It is clear.

4. Some applications of $\mathcal{C}$-continuous, $\mathcal{C}$-closed and $\mathcal{C}$-open functions

In this section we present some applications of these three new type of functions, mainly related to compactness and separability, which constitute generalizations of well-known results of the general topology. If $(X, \tau, \mathcal{I})$ is an ideal space, the set $\overline{\mathcal{I}} = \left\{A \subseteq X : A \subseteq \overline{I}, \text{ for some } I \in \mathcal{I}\right\}$ is an ideal in X. Moreover, it is clear that if $J \in \overline{\mathcal{I}}$ then $\overline{J} \in \overline{\mathcal{I}}$.

Recall that a topological space (X, τ) is said to be quasi-H-closed, or simply QHC [11], if for each open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of X, there exists a finite $\Lambda_0 \subseteq \Lambda$ with $X = \bigcup\limits_{\alpha \in \Lambda_0} \overline{V_\alpha}$. An ideal space $(X, \tau, \mathcal{I})$ is defined to be: (1) $\mathcal{I}$-compact [7] if for all open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of X, there exists a finite $\Lambda_0 \subseteq \Lambda$ such that $X \setminus \bigcup\limits_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{I}$, (2) $\mathcal{I}$-QHC [3] if for all open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of X, there exists a finite $\Lambda_0 \subseteq \Lambda$ such that $X \setminus \bigcup\limits_{\alpha \in \Lambda_0} \overline{V_\alpha} \in \mathcal{I}$, and (3) $\mathcal{I}$-normal

[12] if, for each pair of disjoint closed sets F and G, there are disjoint open sets U and V such that $F \backslash U \in \mathcal{I}$ and $G \backslash V \in \mathcal{I}$.

Theorem 4.1 If $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is sobreyective and $\mathcal{C}$-continuous and if $(X, \tau \oplus \mathcal{I}_{f,\mathcal{J}})$ is compact, then $(Y, \beta, \mathcal{J})$ is $\mathcal{J}$-compact.

Proof. If $\{V_\alpha\}_{\alpha \in \Lambda}$ is an open cover of Y then $X = \bigcup_{\alpha \in \Lambda} f^{-1}(V_\alpha)$. For each $\alpha \in \Lambda$, there exists $J_\alpha \in \mathcal{J}$ such that $f^{-1}(V_\alpha) \subseteq int\left[f^{-1}(V_\alpha)\right] \cup f^{-1}(J_\alpha)$. Thus $X = \bigcup_{\alpha \in \Lambda} \{int\left[f^{-1}(V_\alpha)\right] \cup f^{-1}(J_\alpha)\}$, with $int\left[f^{-1}(V_\alpha)\right] \cup f^{-1}(J_\alpha) \in \tau \oplus \mathcal{I}_{f,\mathcal{J}}$, for each $\alpha \in \Lambda$. There exists a finite $\Lambda_0 \subseteq \Lambda$ such that $X = \bigcup_{\alpha \in \Lambda_0} \{int\left[f^{-1}(V_\alpha)\right] \cup f^{-1}(J_\alpha)\}$. This implies that $X = \left(\bigcup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha)\right) \cup \bigcup_{\alpha \in \Lambda_0} f^{-1}(J_\alpha)$. Hence $Y = \left(\bigcup_{\alpha \in \Lambda_0} V_\alpha\right) \cup \bigcup_{\alpha \in \Lambda_0} J_\alpha$, and so $Y \backslash \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{J}$.

Theorem 4.2 If $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is sobreyective and $\mathcal{C}$-continuous and if $(X, \tau \oplus \mathcal{I}_{f,\mathcal{J}})$ is QHC, then $\left(Y, \beta, \overline{\mathcal{J}}\right)$ is $\overline{\mathcal{J}}-$ QHC.

Proof. Let $\lambda = \tau \oplus \mathcal{I}_{f,\mathcal{J}}$ be. Suppose that $\{V_\alpha\}_{\alpha \in \Lambda}$ is an open cover of Y. For each $\alpha \in \Lambda$ there exists $J_\alpha \in \mathcal{J}$ such that $f^{-1}(V_\alpha) \subseteq f^{-1}(J_\alpha) \cup int_\tau\left(f^{-1}(V_\alpha)\right)$. Hence $X = \bigcup_{\alpha \in \Lambda} \left[f^{-1}(J_\alpha) \cup int_\tau\left(f^{-1}(V_\alpha)\right)\right]$. Given that (X, λ) is QHC, we have that $X = \bigcup_{\alpha \in \Lambda_0} \left[adh_\lambda\left(f^{-1}(J_\alpha)\right) \cup adh_\lambda\left(int_\tau\left(f^{-1}(V_\alpha)\right)\right)\right]$, for some finite $\Lambda_0 \subseteq \Lambda$. But $adh_\lambda(A) \subseteq adh_\tau(A)$, for each $A \subseteq X$. This allows us to conclude that $X = \bigcup_{\alpha \in \Lambda_0} \left[adh_\tau\left(f^{-1}(J_\alpha)\right) \cup adh_\tau\left(f^{-1}(V_\alpha)\right)\right]$. Thus $Y = \bigcup_{\alpha \in \Lambda_0} \left[f\left(adh_\tau\left(f^{-1}(J_\alpha)\right)\right) \cup f\left(adh_\tau\left(f^{-1}(V_\alpha)\right)\right)\right]$. For each $\alpha \in \Lambda_0$ there exist $\{L_\alpha, M_\alpha\} \subseteq \mathcal{J}$ such that $f\left(adh_\tau\left(f^{-1}(V_\alpha)\right)\right) \subseteq adh_\beta\left[f\left(f^{-1}(V_\alpha)\right)\right] \cup L_\alpha = adh_\beta(V_\alpha) \cup L_\alpha$ and $f\left(adh_\tau\left(f^{-1}(J_\alpha)\right)\right) \subseteq adh_\beta\left[f\left(f^{-1}(J_\alpha)\right)\right] \cup M_\alpha = adh_\beta(J_\alpha) \cup M_\alpha$. In consequence, $Y = \bigcup_{\alpha \in \Lambda_0} adh_\beta(V_\alpha) \cup \bigcup_{\alpha \in \Lambda_0} \left[adh_\beta(J_\alpha) \cup L_\alpha \cup M_\alpha\right]$ and so $Y \backslash \bigcup_{\alpha \in \Lambda_0} adh_\beta(V_\alpha) \in \overline{\mathcal{J}}$.

Theorem 4.3 If $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is $\mathcal{C}$-continuous then:
(1) If $\{x_n\}_{n=1}^{\infty}$ is a succession in X, $a \in X$ and $x_n \to a$, then either $\{f(a)\} \in \mathcal{J}$ or $f(x_n) \to f(a)$.
(2) If $\mathcal{F}$ is a filter in X, $a \in X$ and $\mathcal{F} \to a$ then either $\{f(a)\} \in \mathcal{J}$ or $f(\mathcal{F}) \to f(a)$.

Proof.

(1) Suppose that $\{f(a)\} \notin \mathcal{J}$. If $W \in \beta$ and $f(a) \in W$, there exists $J \in \mathcal{J}$ such that $a \in f^{-1}(W) \subseteq f^{-1}(J) \cup int\left(f^{-1}(W)\right)$. Since $\{f(a)\} \notin \mathcal{J}$ then $a \in int\left(f^{-1}(W)\right)$, and so there is a $N \in \mathbf{Z}^+$ such that, if $n \geq N$ then $x_n \in int\left(f^{-1}(W)\right)$. Thus, if $n \geq N$ we have that $f(x_n) \in W$.

(2) Suppose that $\{f(a)\} \notin \mathcal{J}$. If $W \in \beta$ and $f(a) \in W$, there exists $J \in \mathcal{J}$ such that $a \in f^{-1}(W) \subseteq f^{-1}(J) \cup int\left(f^{-1}(W)\right)$. Since $\{f(a)\} \notin \mathcal{J}$ then $a \in int\left(f^{-1}(W)\right)$, and so $int\left(f^{-1}(W)\right) \in \mathcal{F}$. This implies that $f\left(int\left(f^{-1}(W)\right)\right) \in f(\mathcal{F})$. Given that $f\left(int\left(f^{-1}(W)\right)\right) \subseteq W$, we have that $W \in f(\mathcal{F})$.

Theorem 4.4 If $f : (X,\tau) \to (Y,\beta,\mathcal{J})$ is continuous, sobreyective and $\mathcal{C}$-closed , and if (X,τ) is normal, then $(Y,\beta,\mathcal{J})$ is $\mathcal{J}$-normal.

Proof. Suppose that F and G are disjoint closed subsets of Y. There exist disjoint $U \in \tau$ and $V \in \tau$, with $f^{-1}(F) \subseteq U$ and $f^{-1}(G) \subseteq V$. Then $f(X\backslash U) \subseteq Y\backslash F$ and $f(X\backslash V) \subseteq Y\backslash G$. Now, $\left[Y\backslash\overline{f(X\backslash U)}\right] \cap \left[Y\backslash\overline{f(X\backslash V)}\right] = Y\backslash\overline{f[X\backslash(U \cap V)]} = \emptyset$. Moreover
$F\backslash\left[Y\backslash\overline{f(X\backslash U)}\right] = \overline{f(X\backslash U)}\backslash(Y\backslash F) \subseteq \overline{f(X\backslash U)}\backslash f(X\backslash U) \in \mathcal{J}$, because f is $\mathcal{C}$-closed. Hence $F\backslash\left[Y\backslash\overline{f(X\backslash U)}\right] \in \mathcal{J}$. Similarly $G\backslash\left[Y\backslash\overline{f(X\backslash V)}\right] \in \mathcal{J}$.

Definition 4.5 The ideal space $(X,\tau,\mathcal{I})$ is said to be separable-$\mathcal{I}$ if there exists a countable $D \subseteq X$ such that $X\backslash\overline{D} \in \mathcal{I}$.

It is noted that separable $\to$ separable-$\mathcal{I}$ and that (X,τ) is separable if and only if $(X,\tau,\{\emptyset\})$ is separable-$\{\emptyset\}$.

Example 4.6

(1) The space $(\mathbf{R},\mathcal{P}(\mathbf{R}),\mathcal{I} = \mathcal{P}(\mathbf{R}))$ is separable-$\mathcal{I}$, but $(\mathbf{R},\mathcal{P}(\mathbf{R}))$ is not separable. (2) Let β be the topology in $\mathbf{R}$ given by: $V \in \beta$ if and only if, for each $r \in \mathbf{R}$, if $r \in V \cap \mathbf{Q}$ then there exists $\epsilon > 0$ such that $(r - \epsilon, r + \epsilon) \subseteq V$. We have that the space $(\mathbf{R},\beta,\mathcal{I} = \mathcal{P}(\mathbf{R}\backslash\mathbf{Q}))$ is separable-$\mathcal{I}$ since $\mathbf{R}\backslash\overline{\mathbf{Q}} = \mathbf{R}\backslash\mathbf{Q} \in \mathcal{I}$. Besides $(\mathbf{R},\beta)$ is not separable. (3) If $\zeta = \{\emptyset\} \cup \{A \subseteq \mathbf{R} : \mathbf{R}\backslash A$ is countable$\}$ then $(\mathbf{R},\zeta,\mathcal{I} = \mathcal{I}_c(\mathbf{R}))$ is not separable-$\mathcal{I}$, because if $A \subseteq \mathbf{R}$ is countable we have that $\overline{A} = A$ and so $\mathbf{R}\backslash\overline{A} \notin \mathcal{I}$.

Theorem 4.7

(1) If $f : (X,\tau) \to (Y,\beta,\mathcal{J})$ is sobreyective and $\mathcal{C}$-continuous, and if (X,τ) is separable then $(Y,\beta,\mathcal{J})$ is separable-$\mathcal{J}$.

(2) If $\mathcal{I}$ is an ideal in X , $f : (X, \tau) \to (Y, \beta, f(\mathcal{I}))$ is sobreyective and $\mathcal{C}$ -continuous, and if $(X, \tau, \mathcal{I})$ is separable-$\mathcal{I}$, then $(Y, \beta, f(\mathcal{I}))$ is separable-$f(\mathcal{I})$.

(3) If $(X, \tau, \mathcal{I})$ is separable-$\mathcal{I}$ and $V \in \tau$, then $(V, \tau_V, \mathcal{I}_V)$ is separable-$\mathcal{I}_V$.

(4) If $(X, \tau, \mathcal{I})$ is separable-$\mathcal{I}$ and $(Y, \beta, \mathcal{J})$ is separable-$\mathcal{J}$ then the space $(X \times Y, \tau \times \beta, \mathcal{I} \otimes \mathcal{J})$ is separable-$(\mathcal{I} \otimes \mathcal{J})$.

Proof.

(1) There exists $D \subseteq X$, countable, with $\overline{D} = X$. Hence $Y \backslash \overline{f(D)} = f(\overline{D}) \backslash \overline{f(D)} \in \mathcal{J}$, because f is $\mathcal{C}$-continuous. Moreover $f(D)$ is countable.

(2) There is a countable $D \subseteq X$ with $X \backslash \overline{D} \in \mathcal{I}$. Given that $Y \backslash f(\overline{D}) \subseteq f(X \backslash \overline{D}) \in f(\mathcal{I})$ and $f(\overline{D}) \backslash \overline{f(D)} \in f(\mathcal{I})$, then $Y \backslash \overline{f(D)} \in f(\mathcal{I})$.

(3) There exists a countable $D \subseteq X$ such that $X \backslash \overline{D} \in \mathcal{I}$.

Since $V \backslash adh_{\tau_V}(V \cap D) = V \backslash \left[\overline{V \cap D} \cap V \right] = V \cap (X \backslash \overline{D})$ and $X \backslash \overline{D} \in \mathcal{I}$, we have that $V \backslash adh_{\tau_V}(V \cap D) \in \mathcal{I}_V$.

(4) There are countable sets $D \subseteq X$ and $E \subseteq Y$ such that $X \backslash \overline{D} \in \mathcal{I}$ and $Y \backslash \overline{E} \in \mathcal{J}$.

Given that $(X \times Y) \backslash \overline{D \times E} = (X \times Y) \backslash \left(\overline{D} \times \overline{E} \right) = \left[\left(X \backslash \overline{D} \right) \times Y \right] \cup \left[X \times \left(Y \backslash \overline{E} \right) \right]$, we have that $(X \times Y) \backslash \overline{D \times E} \in \mathcal{I} \otimes \mathcal{J}$.

Definition 4.8 The space $(X, \tau, \mathcal{I})$ is said to be regular-$\mathcal{I}$ if for each $U \in \tau$ and $x \in U$, it is true that either $\{x\} \in \mathcal{I}$ or there is a $V \in \tau$ such that $x \in V$ and $\overline{V} \subseteq U$.

It is clear that regular $\to$ regular-$\mathcal{I}$ and that (X, τ) is regular if and only if $(X, \tau, \{\emptyset\})$ is regular-$\{\emptyset\}$.

Example 4.9

(1) Let λ be the topology in $\mathbf{R}$ in which the nboods of any nonzero point being as in the usual topology, while nboods of 0 will have the form $U \backslash \{1, 1/2, 1/3, ...\}$ where U is a nbood of 0 in that usual topology. It is known that $\{1, 1/2, 1/3, ...\}$ is a closed set and that $(\mathbf{R}, \lambda)$ is not regular. However $(\mathbf{R}, \lambda, \mathcal{I} = \mathcal{P}(\{0\}))$ is regular-$\mathcal{I}$. Indeed, suppose that $U \in \lambda$ and $a \in U$.

(i) If $a = 0$ then $\{a\} \in \mathcal{I}$.

(ii) If $a \neq 0$, there is $r \in (0, |a|)$ such that $(a - r, a + r) \subseteq U$. It is very easy to see that $adh_\lambda((a - r/2, a + r/2)) = [a - r/2, a + r/2] \subseteq U$.

(2) The space $(\mathbf{R}, \lambda, \mathcal{I} = \mathcal{P}(\{1\}))$ is not regular-$\mathcal{I}$, because we have that $0 \in \mathbf{R} \backslash \{1, 1/2, 1/3, ...\}$, $\{0\} \notin \mathcal{I}$ and there is no $V \in \lambda$ with $0 \in V \subseteq \overline{V} \subseteq \mathbf{R} \backslash \{1, 1/2, 1/3, ...\}$.

(3) If $\mathcal{I}$ is an admissible ideal in X, then each space $(X, \tau, \mathcal{I})$ is regular-$\mathcal{I}$.

Theorem 4.10

(1) If $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is sobreyective, open, closed and $\mathcal{C}$-continuous, and if (X, τ) is regular then $(Y, \beta, \mathcal{J})$ is regular-$\mathcal{J}$.

(2) If $\mathcal{I}$ is an ideal in X, $f : (X, \tau) \to (Y, \beta, f(\mathcal{I}))$ is sobreyective, open, closed and $\mathcal{C}$-continuous, and if $(X, \tau, \mathcal{I})$ is regular-$\mathcal{I}$ then $(Y, \beta, f(\mathcal{I}))$ is regular-$f(\mathcal{I})$.

(3) If $(X, \tau, \mathcal{I})$ is regular-$\mathcal{I}$ and $(Y, \beta, \mathcal{J})$ is regular-$\mathcal{J}$ then the space $(X \times Y, \tau \times \beta, \mathcal{I} \otimes \mathcal{J})$ is regular-$(\mathcal{I} \otimes \mathcal{J})$.

Proof.

(1) If $W \in \beta$ and $y_0 \in W$, there is a $x_0 \in X$ such that $f(x_0) = y_0$. There exists $J \in \mathcal{J}$ with $x_0 \in f^{-1}(W) \subseteq int(f^{-1}(W)) \cup f^{-1}(J)$. If $x_0 \in f^{-1}(J)$ then $\{y_0\} \in \mathcal{J}$. If $x_0 \in int(f^{-1}(W))$, there is a $U \in \tau$ such that $x_0 \in U$ and $\overline{U} \subseteq int(f^{-1}(W))$. In this case $y_0 \in f(U)$ and $\overline{f(U)} \subseteq f(\overline{U}) \subseteq f(int(f^{-1}(W))) \subseteq f((f^{-1}(W))) = W$, with $f(U) \in \beta$.

(2) Suppose that $W \in \beta$ and $f(x_0) \in W$. There exists $I \in \mathcal{I}$ such that $x_0 \in f^{-1}(W) \subseteq int(f^{-1}(W)) \cup f^{-1}(f(I))$, since f is $\mathcal{C}$-continuous. If $x_0 \in f^{-1}(f(I))$ then $\{f(x_0)\} \in f(\mathcal{I})$. If $x_0 \in int(f^{-1}(W))$ then either $\{x_0\} \in \mathcal{I}$, and so $\{f(x_0)\} \in f(\mathcal{I})$, or there exists $U \in \tau$ such that $x_0 \in U \subseteq \overline{U} \subseteq int(f^{-1}(W))$, and so $f(x_0) \in f(U) \subseteq \overline{f(U)} \subseteq f(\overline{U}) \subseteq f(int(f^{-1}(W))) \subseteq W$, with $f(U) \in \beta$.

(3) Suppose that $W \in \tau \times \beta$ and $(x, y) \in W$. There are $U \in \tau$ and $V \in \beta$ such that $(x, y) \in U \times V \subseteq W$. We have that either $\{x\} \in \mathcal{I}$ or there is a $U_1 \in \tau$ with $x \in U_1$ and $\overline{U_1} \subseteq U$. Moreover, either $\{y\} \in \mathcal{J}$ or there is a $V_1 \in \beta$ with $y \in V_1$ and $\overline{V_1} \subseteq V$.

(a) If $\{x\} \in \mathcal{I}$ or $\{y\} \in \mathcal{J}$, it is clear that $\{(x, y)\} = \{x\} \times \{y\} \in \mathcal{I} \otimes \mathcal{J}$.

(b) If there are $U_1 \in \tau$ and $V_1 \in \beta$, such that $x \in U_1$, $\overline{U_1} \subseteq U$, $y \in V_1$ and $\overline{V_1} \subseteq V$, then $(x, y) \in U_1 \times V_1$ and $\overline{U_1 \times V_1} = \overline{U_1} \times \overline{V_1} \subseteq U \times V \subseteq W$.

Theorem 4.11 If (X, τ) is T_2 and if $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is biyective and $\mathcal{C}$-open, then $(Y, \beta \oplus \mathcal{J})$ is T_2.

Proof. Suppose that $\{u, v\} \subseteq Y$ and $u \neq v$. There exists $\{a, b\} \subseteq X$ with $f(a) = u$ and $f(b) = v$. Now, there is a $\{U, V\} \subseteq \tau$ such that $a \in U$, $b \in V$

and $U \cap V = \emptyset$. Thus $u \in f(U)$, $v \in f(V)$, $f(U) \cap f(V) = \emptyset$, with $f(U)$ and $f(V)$ open-$\mathcal{J}$. So that $\{f(U), f(V)\} \subseteq \beta \oplus \mathcal{J}$.

5. $\mathcal{D}$-Continuous functions

Other forms of continuity via ideals were defined and studied by Abd El Monsef et al. and by Kaniewski et al. In both cases the considered ideal is defined in the domain of the function. Under the same hypothesis, in this section we define the $\mathcal{D}$-continuous functions, a new and natural generalization of the concept of continuity, which turns out to be independent of the concepts introduced by these authors.

Recall that if $\mathcal{I}$ is an ideal in X then a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \beta)$ is said to be $\mathcal{I}$-continuous [1] [El-Monsef form] if, for each $V \in \beta$, $f^{-1}(V)$ is $\mathcal{I}$-open, that is, $f^{-1}(V) \subseteq int\left[\left(f^{-1}(V)\right)^*\right]$. On the other hand, f is defined to be $\mathcal{I}$-continuous [6] [Kaniewski form] if, for each $V \in \beta$, there are $U \in \tau$ and $I \in \mathcal{I}$, such that $f^{-1}(V) = U \backslash I$.

Definition 5.1 If $\mathcal{I}$ is an ideal in X, a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \beta)$ is said to be $\mathcal{D}$ -continuous if, for each $V \in \beta$, we have that $f^{-1}(V)$ is open-$\mathcal{I}$, that is, $f^{-1}(V) \backslash int\left(f^{-1}(V)\right) \in \mathcal{I}$.

It is observed that if f is continuous then f is $\mathcal{D}$-continuous, and that $f : (X, \tau) \rightarrow (Y, \beta)$ is continuous if and only if $f : (X, \tau, \{\emptyset\}) \rightarrow (Y, \beta)$ is $\mathcal{D}$-continuous.

The following theorem establishes a relationship between $\mathcal{C}$-continuity and $\mathcal{D}$-continuity.

Theorem 5.2
(1) The function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \beta)$ is $\mathcal{D}$ -continuous if and only $f^{-1}(F)$ is closed-$\mathcal{I}$, for each closed set $F \subseteq Y$.
(2) If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \beta)$ is a $\mathcal{D}$ -continuous function, then the function $f : (X, \tau) \rightarrow (Y, \beta, f(\mathcal{I}))$ is $\mathcal{C}$ -continuous.

Proof.
(1) It is clear.
(2) If $V \in \beta$ there is a $I \in \mathcal{I}$ such that $f^{-1}(V) = int\left(f^{-1}(V)\right) \cup I \subseteq int\left(f^{-1}(V)\right) \cup f^{-1}(f(I))$, with $f(I) \in f(\mathcal{I})$.

Corollary 2.4 implies that $f : (X, \tau) \rightarrow (Y, \beta, f(\mathcal{I}))$ is $\mathcal{C}$-continuous.

Example 5.3

(1) If $\mathcal{U}$ is the usual topology in $\mathbf{R}$, the identity function $f : (\mathbf{R},\mathcal{U},\mathcal{I} = \mathcal{P}(\mathbf{R})) \rightarrow (\mathbf{R},\mathcal{P}(\mathbf{R}))$ is $\mathcal{D}$-continuous, but f is not continuous.

(2) The function $f : (\mathbf{R},\mathcal{U},\mathcal{I} = \mathcal{P}(\mathbf{Z})) \rightarrow (\mathbf{R},\mathcal{U})$ given by $f(x) = [x]$, where $[x]$ is the integer part of x, is $\mathcal{D}$-continuous, because if $B \subseteq \mathbf{R}$ then $f^{-1}(B) = \bigcup_{n \in B \cap \mathbf{Z}} [n, n+1)$, and given that $\{[n, n+1) : n \in B \cap \mathbf{Z}\}$ is locally finite in $(\mathbf{R},\mathcal{U})$, we have that $\overline{\bigcup_{n \in B \cap \mathbf{Z}} [n, n+1)} = \bigcup_{n \in B \cap \mathbf{Z}} \overline{[n, n+1)} = \bigcup_{n \in B \cap \mathbf{Z}} [n, n+1]$, and so $\overline{f^{-1}(B)}\backslash f^{-1}(B) = \bigcup_{n \in B \cap \mathbf{Z}} [n, n+1] \backslash \bigcup_{n \in B \cap \mathbf{Z}} [n, n+1) \subseteq \bigcup_{n \in B \cap \mathbf{Z}} ([n, n+1] \backslash [n, n+1)) = \{n + 1 : n \in B \cap \mathbf{Z}\} \in \mathcal{I}$.

(3) The function $f : (\mathbf{R},\mathcal{U},\mathcal{I} = \mathcal{P}(\{-1,1\})) \rightarrow (\mathbf{R},\mathcal{U})$ given by $f(x) = -1$ if $x < 0$, and $f(x) = 1$ if $x \geq 0$, is not $\mathcal{D}$-continuous since if $A = \{-1\}$ then $A = \overline{A}$ and $\overline{f^{-1}(A)}\backslash f^{-1}\left(\overline{A}\right) = \{0\} \notin \mathcal{I}$. However, $f : (\mathbf{R},\mathcal{U}) \rightarrow (\mathbf{R},\mathcal{U}, f(\mathcal{I}))$ is $\mathcal{C}$-continuous given that $f(\mathcal{I}) = \mathcal{I}$ and if $A \subseteq \mathbf{R}$ then $f\left(\overline{A}\right)\backslash \overline{f(A)} \subseteq \{-1,1\}$ and so $f\left(\overline{A}\right)\backslash \overline{f(A)} \in f(\mathcal{I})$.

In the three next theorems we present other characterizations of $\mathcal{D}$-continuity in terms of interior, adherence and frontier operators.

Theorem 5.4 The following propositions are equivalents:
(1) The function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \beta)$ is $\mathcal{D}$-continuous.
(2) For each $B \subseteq Y$, $f^{-1}(B^{\circ})\backslash int\left(f^{-1}(B)\right) \in \mathcal{I}$.
(3) For each $B \subseteq Y$, $\overline{f^{-1}(B)}\backslash f^{-1}\left(\overline{B}\right) \in \mathcal{I}$.

Proof. It is easy to be established.

Theorem 5.5 The function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \beta)$ is $\mathcal{D}$-continuous if and only if $Fr\left(f^{-1}(B)\right)\backslash f^{-1}\left(Fr(B)\right) \in \mathcal{I}$, for each $B \subseteq Y$.

Proof. $(\rightarrow)$ If $B \subseteq Y$, there exists $\{I_1, I_2\} \subseteq \mathcal{I}$ such that $\overline{f^{-1}(B)} \subseteq f^{-1}\left(\overline{B}\right) \cup I_1$ and $\overline{f^{-1}(Y\backslash B)} \subseteq f^{-1}\left(\overline{Y\backslash B}\right) \cup I_2$, by Theorem 5.4. Then $Fr\left(f^{-1}(B)\right) = \overline{f^{-1}(B)} \cap \overline{f^{-1}(Y\backslash B)} \subseteq \left[f^{-1}\left(\overline{B}\right) \cup I_1\right] \cap \left[f^{-1}\left(\overline{Y\backslash B}\right) \cup I_2\right] = f^{-1}(Fr(B)) \cup I$, where $I = \left[f^{-1}\left(\overline{B}\right) \cap I_2\right] \cup \left[I_1 \cap f^{-1}\left(\overline{Y\backslash B}\right)\right] \cup (I_1 \cap I_2) \in \mathcal{I}$. Hence $Fr\left(f^{-1}(B)\right)\backslash f^{-1}(Fr(B)) \in \mathcal{I}$.

$(\leftarrow)$ If $F \subseteq Y$ is closed, there is a $I \in \mathcal{I}$ such that $Fr\left(f^{-1}(F)\right) \subseteq f^{-1}(Fr(F)) \cup I \subseteq f^{-1}(F) \cup I$, and so $\overline{f^{-1}(F)}\backslash int\left(f^{-1}(F)\right) \subseteq f^{-1}(F) \cup I$.

Thus $\overline{f^{-1}(F)} \subseteq int\left(f^{-1}(F)\right) \cup f^{-1}(F) \cup I = f^{-1}(F) \cup I$. This implies that $\overline{f^{-1}(F)}\backslash f^{-1}(F) \in \mathcal{I}$. By Theorem 5.2 f is $\mathcal{D}$-continuous.

Theorem 5.6 The function $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is $\mathcal{D}$-continuous if and only if, for each $A \subseteq X$, there is a $I \in \mathcal{I}$ such that $f\left(\overline{A}\backslash I\right) \subseteq \overline{f(A)}$.

Proof. $(\to)$ If $A \subseteq X$, there is a $I \in \mathcal{I}$ such that $\overline{f^{-1}(f(A))}\backslash f^{-1}\left(\overline{f(A)}\right) = I$, by Theorem 5.4. Hence $\overline{A}\backslash I \subseteq f^{-1}\left(\overline{f(A)}\right)$, and so $f\left(\overline{A}\backslash I\right) \subseteq \overline{f(A)}$.

$(\leftarrow)$ If $B \subseteq Y$ then there exists $I \in \mathcal{I}$ with $f\left(\overline{f^{-1}(B)}\backslash I\right) \subseteq \overline{f\left(f^{-1}(B)\right)} \subseteq \overline{B}$. This allows us to conclude that $\overline{f^{-1}(B)}\backslash f^{-1}\left(\overline{B}\right) \in \mathcal{I}$. By Theorem 5.4, f is $\mathcal{D}$-continuous.

Corollary 5.7 If the function $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is $\mathcal{D}$-continuous then $f\left(\overline{\mathcal{I}}\right) \subseteq \overline{f(\mathcal{I})}$.

Proof. Let $J \in \overline{\mathcal{I}}$ be. There exists $I \in \mathcal{I}$ such that $J \subseteq \overline{I}$. There is a $I_1 \in \mathcal{I}$ with $f\left(\overline{I}\backslash I_1\right) \subseteq \overline{f(I)}$, by Theorem 5.6. Since we also have that $f\left(\overline{I}\backslash\right)\backslash f(I_1) \subseteq f\left(\overline{I}\backslash I_1\right)$, then $f(J) \subseteq f\left(\overline{I}\right) \subseteq \overline{f(I)} \cup f(I_1) \subseteq \overline{f(I \cup I_1)} \in \overline{f(\mathcal{I})}$.

Theorem 5.8

(1) If $\{\mathcal{I}_\alpha\}_{\alpha \in \Lambda}$ is a collection of ideals in X and if $\mathcal{I} = \bigcap_{\alpha \in \Lambda} \mathcal{I}_\alpha$, then $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is $\mathcal{D}$-continuous if and only if $f : (X, \tau, \mathcal{I}_\alpha) \to (Y, \beta)$ is $\mathcal{D}$-continuous, for each $\alpha \in \Lambda$.

(2) If $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is $\mathcal{D}$-continuous and $g : (Y, \beta) \to (Z, \gamma)$ is continuous, then $g \circ f$ is $\mathcal{D}$-continuous.

(3) If $A \subseteq X$ then the function $\chi_A : (X, \tau, \mathcal{I}) \to (\{0,1\}, \mathcal{P}(\{0,1\}))$ is $\mathcal{D}$-continuous if and only if $Fr(A) \in \mathcal{I}$.

(4) If A and B are open sets in (X, τ), $X = A \cup B$ and if $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is a function such that the restrictions $f_A : (A, \tau_A, \mathcal{I}_A) \to (Y, \beta)$ and $f_B : (B, \tau_B, \mathcal{I}_B) \to (Y, \beta)$ are $\mathcal{D}$-continuous, then f is $\mathcal{D}$-continuous.

(5) If $A \subseteq X$ and if $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is $\mathcal{D}$-continuous, then $f_A : (A, \tau_A, \mathcal{I}_A) \to (Y, \beta)$ is $\mathcal{D}$-continuous.

Proof. (3) $(\to)$ Given that $\chi_A^{-1}\{0\}$ and $\chi_A^{-1}\{1\}$ are open-$\mathcal{I}$, then $Fr(A) = \overline{A}\backslash A^\circ = \left(\overline{A}\backslash A\right) \cup (A\backslash A^\circ) \in \mathcal{I}$. $(\leftarrow)$ If $Fr(A) \in \mathcal{I}$ then $\overline{A}\backslash A \in \mathcal{I}$ and $A\backslash A^\circ \in \mathcal{I}$ or, equivalently, $\chi_A^{-1}\{0\}$ and $\chi_A^{-1}\{1\}$ are open-$\mathcal{I}$.

(4) Suppose that $V \in \beta$. Given that $\{A, B\} \subseteq \tau$, $int_{\tau_A}\left(f_A^{-1}(V)\right) = A \cap int\left(f^{-1}(V)\right)$ and $int_{\tau_B}\left(f_B^{-1}(V)\right) = B \cap int\left(f^{-1}(V)\right)$.

Since $f^{-1}(V) = f_A^{-1}(V) \cup f_B^{-1}(V)$, we have that $f^{-1}(V) \setminus int\left(f^{-1}(V)\right) \subseteq \left[f_A^{-1}(V) \setminus int_{\tau_A}\left(f_A^{-1}(V)\right)\right] \cup \left[f_B^{-1}(V) \setminus int_{\tau_B}\left(f_B^{-1}(V)\right)\right] \in \mathcal{I}_A \cup \mathcal{I}_B \subseteq \mathcal{I}$.

The other parts of the theorem are obtained without any problem.

Theorem 5.9 If $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ and $g : (Y, \beta, \mathcal{L}) \to (Z, \gamma)$ are $\mathcal{D}$-continuous and if $\left\{f^{-1}(L) : L \in \mathcal{L}\right\} \subseteq \mathcal{I}$, then $g \circ f$ is $\mathcal{D}$-continuous.

Proof. If $W \in \gamma$ there exists $L \in \mathcal{L}$ such that $g^{-1}(W) = int\left(g^{-1}(W)\right) \cup L$. Now, there is a $I \in \mathcal{I}$ with $f^{-1}\left[int\left(g^{-1}(W)\right)\right] = int\left\{f^{-1}\left[int\left(g^{-1}(W)\right)\right]\right\} \cup I$. So $f^{-1}\left(g^{-1}(W)\right) = f^{-1}\left(int\left(g^{-1}(W)\right)\right) \cup f^{-1}(L) = int\left\{f^{-1}\left[int\left(g^{-1}(W)\right)\right]\right\} \cup I \cup f^{-1}(L) \subseteq int\left\{f^{-1}\left(g^{-1}(W)\right)\right\} \cup I \cup f^{-1}(L)$.

Hence $(g \circ f)^{-1}(W) \setminus int\left[(g \circ f)^{-1}(W)\right] \subseteq I \cup f^{-1}(L) \in \mathcal{I}$.

Corollary 5.10 If $f : X \to Y$ is one-one and if $g : (Y, \beta, \mathcal{L}) \to (Z, \gamma)$ and $f : (X, \tau, f^{-1}(\mathcal{L})) \to (Y, \beta)$ are $\mathcal{D}$-continuous, then $g \circ f$ is $\mathcal{D}$-continuous.

The composition of $\mathcal{D}$-continuous functions may not be a $\mathcal{D}$-continuous function, as we show in the following example.

Example 5.11 Suppose that $X = Y = Z = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$, $\beta = \{\emptyset, \{a, c\}, Y\}$, $\gamma = \{\emptyset, \{c\}, Z\}$, $\mathcal{I} = \mathcal{J} = \mathcal{P}(\{b, c\})$ and that $f : X \to Y$, $g : Y \to Z$ are defined by: $f(a) = b$, $f(b) = a$, $f(c) = c$, $g(a) = b$, $g(b) = c$ and $g(c) = a$. It is not a problem to prove that f and g are $\mathcal{D}$-continuous. Given that $(g \circ f)^{-1}(\{c\}) \setminus int\left[(g \circ f)^{-1}(\{c\})\right] = \{a\} \notin \mathcal{I}$, we conclude that $g \circ f$ is not $\mathcal{D}$-continuous.

Theorem 5.12 If $\mathcal{I}$ is an ideal in X , then $f : (X, \tau, \mathcal{I}^{\otimes}) \to (Y, \beta)$ is $\mathcal{D}$-continuous if and only if $f : (X, \tau \oplus \mathcal{I}) \to (Y, \beta)$ is continuous.

Proof. $(\to)$ If $V \in \beta$, there is a $J \in \mathcal{I}^{\otimes}$ such that $f^{-1}(V) \setminus int\left(f^{-1}(V)\right) = J$. So that $f^{-1}(V) = int\left(f^{-1}(V)\right) \cup J \in \tau \oplus \mathcal{I}$.

$(\leftarrow)$ If $W \in \beta$, there are $U \in \tau$ and $J \in \mathcal{I}^{\otimes}$ such that $f^{-1}(W) = U \cup J$. Then $U \subseteq int_\tau\left(f^{-1}(W)\right)$ and so $f^{-1}(W) \subseteq int_\tau\left(f^{-1}(W)\right) \cup J$. This implies that $f^{-1}(W) \setminus int_\tau\left(f^{-1}(W)\right) \subseteq J$. Thus $f^{-1}(W) \setminus int_\tau\left(f^{-1}(W)\right) \in \mathcal{J}^{\otimes}$.

It is possible to build new $\mathcal{D}$-continuous functions from some previously known ones, as we will show now.

Theorem 5.13
(1) If $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ and $g : (X, \tau, \mathcal{I}) \to (Z, \gamma)$ are $\mathcal{D}$-continuous, then the function $h : (X, \tau, \mathcal{I}^{\otimes}) \to (Y \times Z, \beta \times \gamma)$, defined by $h(x) = (f(x), g(x))$, is $\mathcal{D}$-continuous.
(2) If $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is $\mathcal{D}$-continuous then the function $g : (X, \tau \oplus \mathcal{I}) \to (X \times Y, (\tau \oplus \mathcal{I}) \times \beta)$ defined by $g(x) = (x, f(x))$, is continuous. Moreover, if g is continuous then $f : (X, \tau, \mathcal{I}^{\otimes}) \to (Y, \beta)$ is $\mathcal{D}$-continuous.

Proof.
(1) If $W \in \beta \times \gamma$, there are $\{U_j : j \in \Lambda\} \subseteq \beta$ and $\{V_j : j \in \Lambda\} \subseteq \gamma$ such that $W = \bigcup_{j \in \Lambda} (U_j \times V_j)$. Hence $h^{-1}(W) = \bigcup_{j \in \Lambda} h^{-1}(U_j \times V_j) = \bigcup_{j \in \Lambda} f^{-1}(U_j) \cap g^{-1}(V_j)$.
For each $j \in \Lambda$ there exists $\{I_j, J_j\} \subseteq \mathcal{I}$ such that $f^{-1}(U_j) = int\left[f^{-1}(U_j)\right] \cup I_j$ and $g^{-1}(V_j) = int\left[g^{-1}(V_j)\right] \cup J_j$.
Thus $h^{-1}(W) = \bigcup_{j \in \Lambda} \left[int\left(f^{-1}(U_j)\right) \cup I_j\right] \cap \left[int\left(g^{-1}(V_j)\right) \cup J_j\right] \subseteq int\left(h^{-1}(W)\right) \cup I$, where $I = \bigcup_{j \in \Lambda} \left[int\left(f^{-1}(U_j)\right) \cap J_j\right] \cup \left[int\left(g^{-1}(V_j)\right) \cap I_j\right] \cup (I_j \cap J_j) \in \mathcal{I}^{\otimes}$.
Hence $h^{-1}(W) \setminus int\left[h^{-1}(W)\right] \in \mathcal{I}^{\otimes}$.
(2) If $U \in \tau$ and $V \in \beta$, $g^{-1}(U \times V) = U \cap f^{-1}(V) \in \tau \oplus \mathcal{I}$. Now, if $I \in \mathcal{I}$ and $V \in \beta$, $g^{-1}(I \times V) = I \cap f^{-1}(V) \in \mathcal{I} \subseteq \tau \oplus \mathcal{I}$. This implies that g is continuous.
Finally, if $V \in \beta$ then $f^{-1}(V) = g^{-1}(X \times V) \in \tau \oplus \mathcal{I}$, and so $f^{-1}(V)$ is open-$\mathcal{I}^{\otimes}$.

Theorem 5.14
(1) If $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is one-one then f is $\mathcal{D}$-continuous if and only if $f : (X, \tau) \to (Y, \beta, f(\mathcal{I}))$ is $\mathcal{C}$-continuous.
(2) If $f : X \to Y$ is one-one, then $f : (X, \tau) \to (Y, \beta, \mathcal{J})$ is $\mathcal{C}$-continuous if and only if $f : (X, \tau, f^{-1}(\mathcal{J})) \to (Y, \beta)$ is $\mathcal{D}$-continuous.

Proof. It is easy to be established.

In our final property for this new type of weak continuity, we present the smallest ideal $\mathcal{I}$ in X for which a given function $f : (X, \tau) \to (Y, \beta)$ is $\mathcal{D}$-continuous.

Theorem 5.15 If $f : (X, \tau) \to (Y, \beta)$ is a function and if

$$\mathcal{I}_f = \left\{ A \subseteq X : \text{there is a finite } \beta_A \subseteq \beta \text{ with } A \subseteq \bigcup_{V \in \beta_A} \left[f^{-1}(V) \setminus int\left(f^{-1}(V) \right) \right] \right\},$$

then $\mathcal{I}_f$ is the smallest ideal $\mathcal{I}$ in X such that $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is $\mathcal{D}$-continuous.

Proof. It is clear that $\mathcal{I}_f$ is an ideal in X. Given that $f^{-1}(V) \setminus int\left(f^{-1}(V) \right) \in \mathcal{I}_f$, for each $V \in \beta$, then $f : (X, \tau, \mathcal{I}_f) \to (Y, \beta)$ is $\mathcal{D}$-continuous. Now, suppose that $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is $\mathcal{D}$-continuous. If $A \in \mathcal{I}_f$ then there is a finite $\beta_A \subseteq \beta$ with $A \subseteq \bigcup_{V \in \beta_A} \left[f^{-1}(V) \setminus int\left(f^{-1}(V) \right) \right]$. Since $f^{-1}(V) \setminus int\left(f^{-1}(V) \right) \in \mathcal{I}$, for each $V \in \beta_A$, we can conclude that $A \in \mathcal{I}$. In this way $\mathcal{I}_f \subseteq \mathcal{I}$.

In the next example we show that $\mathcal{D}$-continuous and $\mathcal{I}$-continuous (Kaniewski form) are independent concepts.

Example 5.16
(1) Consider the function $f : (\mathbf{R}, \mathcal{U}, \mathcal{I} = \mathcal{P}(\mathbf{Z})) \to (\mathbf{R}, \mathcal{L})$ defined by $f(x) = [x]$. Here $\mathcal{L}$ is the Sorgenfrey topology in $\mathbf{R}$. If $V \in \mathcal{L}$ then $f^{-1}(V) = \bigcup_{n \in V \cap \mathbf{Z}} [n, n+1)$ and $\bigcup_{n \in V \cap \mathbf{Z}} (n, n+1) \subseteq int\left(f^{-1}(V) \right)$. Therefore $f^{-1}(V) \setminus int\left(f^{-1}(V) \right) \subseteq \mathbf{Z}$. Hence $f^{-1}(V)$ is open-$\mathcal{I}$, and f is $\mathcal{D}$-continuous.

On the other hand, if $V = [0, 1)$, we have that $[0, 1) \in \mathcal{L}$ but there are no $U \in \mathcal{U}$ and $I \in \mathcal{I}$ such that $f^{-1}(V) = U \setminus I$, because $f^{-1}(V) = [0, 1)$. In effect, if $U \in \mathcal{U}$, $I \in \mathcal{I}$ and $f^{-1}(V) = U \setminus I$, we have that $U = [0, 1) \cup (U \cap I)$, and so there exists $\varepsilon \in (0, 1)$ such that $(-\varepsilon, 0) \subseteq U \cap I \subseteq \mathbf{Z}$, absurd. In conclusion, f is not $\mathcal{I}$-continuous (Kaniewski form).
(2) Let $\beta = \{\emptyset, \mathbf{R}\} \cup \{(a, \infty) : a \in \mathbf{R}\}$ and $g : (\mathbf{R}, \mathcal{U}, \mathcal{I} = \mathcal{P}(\mathbf{R} \setminus \mathbf{Z})) \to (\mathbf{R}, \beta)$, defined by $g(x) = [x]$. If $r \in \mathbf{R}$ then $g^{-1}((r, +\infty)) = [[r] + 1, +\infty) = (r, +\infty) \setminus I$, where $I = (r, [r] + 1) \in \mathcal{I}$. This implies that g is $\mathcal{I}$-continuous (Kaniewski form).

Now, since $g^{-1}\left(\left(\frac{1}{2}, +\infty \right) \right) \setminus int\left(g^{-1}\left(\left(\frac{1}{2}, +\infty \right) \right) \right) = [1, +\infty) \setminus (1, +\infty) = \{1\} \notin \mathcal{I}$, we conclude that g is not $\mathcal{D}$-continuous.

Finally, we show that $\mathcal{D}$-continuous and $\mathcal{I}$-continuous (El Monsef form) are independent concepts.

Example 5.17
(1) The function f of Example 5.16-(1) is not $\mathcal{I}$-continuous (El Monsef

form) because $\left(f^{-1}\left([0,1)\right)\right)^{*} = [0,1)^{*} = [0,1]$ and $int\left[\left(f^{-1}\left([0,1)\right)\right)^{*}\right] = (0,1)$, and so $f^{-1}\left([0,1)\right)\ int\left[\left(f^{-1}\left([0,1)\right)\right)^{*}\right]$.

(2) If $\lambda = \{\emptyset\} \cup \{A \subseteq \mathbf{R} : \mathbf{R} \backslash A \text{ is finite}\}$ and $\beta = \{\emptyset, \mathbf{R}\} \cup \{(a, \infty) : a \in \mathbf{R}\}$, the function $f : (\mathbf{R}, \lambda, \mathcal{I} = \mathcal{I}_{f}(\mathbf{R})) \rightarrow (\mathbf{R}, \beta)$ defined by $f(x) = [x]$ is $\mathcal{I}$-continuous (El Monsef form) because

$f^{-1}(\mathbf{R}) = \mathbf{R} = int(\mathbf{R}^{*})$, $f^{-1}(\emptyset) = \emptyset = int(\emptyset^{*})$ and, if $a \in \mathbf{R}$, $f^{-1}((a, +\infty)) = [[a] + 1, +\infty) \subseteq int\left[\left(f^{-1}((a, +\infty))\right)^{*}\right] = \mathbf{R}$.

However f is not $\mathcal{D}$-continuous because
$f^{-1}\left(\left(\frac{1}{2}, +\infty\right)\right) \backslash int\left(f^{-1}\left(\left(\frac{1}{2}, +\infty\right)\right)\right) = [1, +\infty) \notin \mathcal{I}$.

6. $\mathcal{D}$-open functions

In this section we introduce an extension of the open functions to the ideal spaces, in the case in which the considered ideal is defined in the domain of the function.

Definition 6.1 The function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \beta)$ is said to be $\mathcal{D}$ -open if, for each $U \in \tau$, there exists $I \in \mathcal{I}$ such that $f(U \backslash I) \subseteq int(f(U))$.

It is noted that if $f : (X, \tau) \rightarrow (Y, \beta)$ is open then $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \beta)$ $\mathcal{D}$-open, and that if $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \beta)$ is $\mathcal{D}$-open then $f : (X, \tau) \rightarrow (Y, \beta, f(\mathcal{I}))$ $\mathcal{C}$-open. The reciprocal implications, in general, are false, as we can see in the following examples.

Examples 6.2
1) The function $f : (\mathbf{R}, \mathcal{U}, \mathcal{I} = \mathcal{P}(\mathbf{R})) \rightarrow (\mathbf{R}, \{\emptyset, \mathbf{R}\})$, given by $f(x) = x$, is $\mathcal{D}$-open but f is not open.
2) Suppose that $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, \{b, c\}, \{b\}, X\}$, $\beta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Let $f : X \rightarrow Y$ be the function defined by $f(a) = b$, $f(b) = c$ and $f(c) = a$. It is not a problem to prove that f is $\mathcal{D}$-open. Now, given that $f(\{a, b\}) = \{b, c\} \notin \beta$ then f is not open.
3) Suppose that X, Y, τ and β are as in example 2, that $\mathcal{I} = \{\emptyset, \{a\}\}$, and that $f : X \rightarrow Y$ is defined by $f(a) = f(b) = c$ and $f(c) = a$. It is observed that $f(\mathcal{I}) = \{\emptyset, \{c\}\}$ and that $f : (X, \tau) \rightarrow (Y, \beta, f(\mathcal{I}))$ is $\mathcal{C}$-open. However f is not $\mathcal{D}$-open since there is no a $I \in \mathcal{I}$ such that $f(\{b, c\} \backslash I) \subseteq int(f(\{b, c\}))$.

Theorem 6.3
(1) If $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is $\mathcal{D}$-open and $g : (Y, \beta) \to (Z, \gamma)$ is open, then $g \circ f$ is $\mathcal{D}$-open.
(2) If $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ and $g : (Y, \beta, \mathcal{J}) \to (Z, \gamma)$ are $\mathcal{D}$-open, $g \circ f$ is one-one and $\mathcal{J} \subseteq f(\mathcal{I})$, then $g \circ f$ is $\mathcal{D}$-open.

Proof.
1) If $U \in \tau$ then there exists $I \in \mathcal{I}$ such that $f(U \backslash I) \subseteq int(f(U))$. Hence $f(U) \subseteq int(f(U)) \cup f(I)$ and $(g \circ f)(U) \subseteq g(int(f(U))) \cup (g \circ f)(I) \subseteq int[(g \circ f)(U)] \cup (g \circ f)(I)$.

2) If $U \in \tau$ then there exists $I \in \mathcal{I}$ such that $f(U \backslash I) \subseteq int(f(U))$. Hence $f(U) \subseteq int(f(U)) \cup f(I)$ and $(g \circ f)(U) \subseteq g(int(f(U))) \cup (g \circ f)(I)$. Now, there exists $J \in \mathcal{J}$ with $g[int(f(U)) \backslash J] \subseteq int[g(int(f(U)))]$. There is a $I_1 \in \mathcal{I}$ such that $J = f(I_1)$. Thus $g(int(f(U))) \cup (g \circ f)(I) \subseteq int[g(int(f(U)))] \cup (g \circ f)(I \cup I_1) \subseteq int[(g \circ f)(U)] \cup (g \circ f)(I \cup I_1)$. Since $g \circ f$ is inyective,
$(g \circ f)(U \backslash (I \cup I_1)) = (g \circ f)(U) \backslash (g \circ f)(I \cup I_1) \subseteq int[(g \circ f)(U)]$.

The composition of $\mathcal{D}$-open functions may not be a $\mathcal{D}$-open function, as we show in the next example.

Example 6.4 Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$, $\beta = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$, $\gamma = \{\emptyset, Z, \{c\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$ and $\mathcal{J} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Consider the functions $f : X \to Y$ and $g : Y \to Z$ defined by: $f(a) = b$, $f(b) = c$, $f(c) = a$, and $g(a) = b$, $g(b) = a$, $g(c) = c$. It is not a problem to verify that f and g are $\mathcal{D}$-open. However, since there is no a $I \in \mathcal{I}$ such that $(g \circ f)[\{a, b\} \backslash I] \subseteq int[(g \circ f)(\{a, b\})]$, we have that $g \circ f$ is not $\mathcal{D}$-open.

7. Some applications of $\mathcal{D}$-continuous, $\mathcal{D}$-closed and $\mathcal{D}$-open functions

In this final section we present some properties of $\mathcal{D}$-continuous and $\mathcal{D}$-open functions, mainly related to compactness and separability. This results also are generalizations of well-known results in general topology.

Recall that an ideal space $(X, \tau, \mathcal{I})$ is: (1) $\rho\mathcal{I}$-compact [9] if, for each collection $\{U_\alpha\}_{\alpha \in \Lambda}$ of open sets, if $X \backslash \bigcup_{\alpha \in \Lambda} U_\alpha \in \mathcal{I}$, there exists a finite $\Lambda_0 \subseteq \Lambda$ such that $X \backslash \bigcup_{\alpha \in \Lambda_0} U_\alpha \in \mathcal{I}$, and (2) $\sigma\mathcal{I}$-compact [9] if, for each

non-empty collection $\{U_\alpha\}_{\alpha\in\Lambda}$ of non-empty open sets, if $X\backslash \bigcup\limits_{\alpha\in\Lambda} U_\alpha \in \mathcal{I}$, there exists a finite $\Lambda_0 \subseteq \Lambda$ such that $X = \bigcup\limits_{\alpha\in\Lambda_0} U_\alpha$.

Theorem 7.1 Suppose that $f : (X,\tau,\mathcal{I}) \to (Y,\beta)$ is sobreyective and $\mathcal{D}$-continuous and that $(X,\tau\oplus\mathcal{I})$ is compact. Then $(Y,\beta,f(\mathcal{I}))$ is $f(\mathcal{I})$-compact.

Proof. If $\{V_\alpha\}_{\alpha\in\Lambda}$ is an open cover of Y, then $X = \bigcup\limits_{\alpha\in\Lambda} f^{-1}(V_\alpha)$. For each $\alpha \in \Lambda$there exists $I_\alpha \in \mathcal{I}$ such that $f^{-1}(V_\alpha) = int_\tau\left(f^{-1}(V_\alpha)\right) \cup I_\alpha$. Then $X = \bigcup\limits_{\alpha\in\Lambda}\left(int_\tau\left(f^{-1}(V_\alpha)\right) \cup I_\alpha\right)$. Given that $(X,\tau\oplus\tau)$ is compact, there is a finite $\Lambda_0 \subseteq \Lambda$ such that $X = \bigcup\limits_{\alpha\in\Lambda_0}\left(int_\tau\left(f^{-1}(V_\alpha)\right) \cup I_\alpha\right)$. Thus $X = \bigcup\limits_{\alpha\in\Lambda_0}\left(f^{-1}(V_\alpha)\cup I_\alpha\right)$, and this implies $Y = \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha \cup \bigcup\limits_{\alpha\in\Lambda_0} f(I_\alpha)$. Hence $Y\backslash \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha \subseteq \bigcup\limits_{\alpha\in\Lambda_0} f(I_\alpha) \in f(\mathcal{I})$.

Theorem 7.2 Suppose that $f : (X,\tau,\mathcal{I}) \to (Y,\beta)$ is sobreyective and $\mathcal{D}$-continuous. Then:
(1) If $(X,\tau,\mathcal{I}^\otimes)$ is $\sigma\mathcal{I}^\otimes$-compact then (Y,β) is compact.
(2) If $(X,\tau,\mathcal{I}^\otimes)$ is $\rho\mathcal{I}^\otimes$-compact then $(Y,\beta,f(\mathcal{I}^\otimes))$ is $f(\mathcal{I}^\otimes)$-compact.
(3) If $(X,\tau\oplus\mathcal{I})$ is QHC then $\left(Y,\beta,f\left(\overline{\mathcal{I}}\right)\right)$ is $f\left(\overline{\mathcal{I}}\right)$-QHC.

Proof.
(1) If $\{V_\alpha\}_{\alpha\in\Lambda}$ is an open cover of Y, then $X = \bigcup\limits_{\alpha\in\Lambda} f^{-1}(V_\alpha)$. For each $\alpha \in \Lambda$ there exists a $I_\alpha \in \mathcal{I}$ such that $f^{-1}(V_\alpha) = int\left(f^{-1}(V_\alpha)\right)\cup I_\alpha$. Then $X\backslash \bigcup\limits_{\alpha\in\Lambda} int\left(f^{-1}(V_\alpha)\right) \subseteq \bigcup\limits_{\alpha\in\Lambda} I_\alpha \in \mathcal{I}^\otimes$. Since $(X,\tau,\mathcal{I}^\otimes)$ is $\sigma\mathcal{I}^\otimes$-compact, there exists a finite $\Lambda_0 \subseteq \Lambda$ such that $X = \bigcup\limits_{\alpha\in\Lambda_0} int\left(f^{-1}(V_\alpha)\right)$. Then $X = \bigcup\limits_{\alpha\in\Lambda_0} f^{-1}(V_\alpha)$, and so $Y = \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha$.
(2) If $\{V_\alpha\}_{\alpha\in\Lambda}$ is an open cover of Y, then $X = \bigcup\limits_{\alpha\in\Lambda} f^{-1}(V_\alpha)$. For each $\alpha \in \Lambda$ there exists $I_\alpha \in \mathcal{I}$ such that $f^{-1}(V_\alpha) = int\left(f^{-1}(V_\alpha)\right)\cup I_\alpha$. Then $X\backslash \bigcup\limits_{\alpha\in\Lambda} int\left(f^{-1}(V_\alpha)\right) \subseteq \bigcup\limits_{\alpha\in\Lambda} I_\alpha \in \mathcal{I}^\otimes$. Since $(X,\tau,\mathcal{I}^\otimes)$ is $\rho\mathcal{I}^\otimes$-compact, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $X\backslash \bigcup\limits_{\alpha\in\Lambda_0} int\left(f^{-1}(V_\alpha)\right) \in \mathcal{I}^\otimes$.

Then $X\backslash \bigcup\limits_{\alpha\in\Lambda_0} f^{-1}(V_\alpha) \in \mathcal{I}^\otimes$. Given that $Y\backslash \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha \subseteq f\left(X\backslash \bigcup\limits_{\alpha\in\Lambda_0} f^{-1}(V_\alpha)\right) \in$

$f(\mathcal{I}^{\otimes})$, we have that $Y \backslash \bigcup_{\alpha \in \Lambda_0} V_\alpha \in f(\mathcal{I}^{\otimes})$.

(3) Let $\lambda = \tau \oplus \mathcal{I}$ be. We will denote $adh_\tau(A)$ by $\overline{A}$, for all $A \subseteq X$. If $\{V_\alpha\}_{\alpha \in \Lambda}$ is an open cover of Y, then $X = \bigcup_{\alpha \in \Lambda} f^{-1}(V_\alpha)$. For each $\alpha \in \Lambda$ there exists a $I_\alpha \in \mathcal{I}$ such that $f^{-1}(V_\alpha) = int_\tau(f^{-1}(V_\alpha)) \cup I_\alpha$. Then $X = \bigcup_{\alpha \in \Lambda} [int_\tau(f^{-1}(V_\alpha)) \cup I_\alpha]$.

Since $(X, \tau \oplus \mathcal{I})$ is a QHC space, there exists a finite $\Lambda_0 \subseteq \Lambda$ such that $X = \bigcup_{\alpha \in \Lambda_0} [adh_\lambda(int_\tau(f^{-1}(V_\alpha))) \cup adh_\lambda(I_\alpha)] \subseteq \bigcup_{\alpha \in \Lambda_0} [adh_\lambda(f^{-1}(V_\alpha)) \cup adh_\lambda(I_\alpha)]$. But $adh_\lambda(A) \subseteq \overline{A}$, for each $A \subseteq X$.

Then $X = \bigcup_{\alpha \in \Lambda_0} \left[\overline{f^{-1}(V_\alpha)} \cup \overline{I_\alpha}\right]$, and so $Y = \bigcup_{\alpha \in \Lambda_0} f\left[\overline{f^{-1}(V_\alpha)}\right] \cup \bigcup_{\alpha \in \Lambda_0} f\left(\overline{I_\alpha}\right)$. For each $\alpha \in \Lambda_0$ there exists $J_\alpha \in \mathcal{I}$ with $\overline{f^{-1}(V_\alpha)} \backslash f^{-1}(adh_\beta(V_\alpha)) = J_\alpha$. Hence $Y = \bigcup_{\alpha \in \Lambda_0} f\left(\overline{I_\alpha}\right) \cup \bigcup_{\alpha \in \Lambda_0} f[f^{-1}(adh_\beta(V_\alpha))] \cup \bigcup_{\alpha \in \Lambda_0} f(J_\alpha) = \bigcup_{\alpha \in \Lambda_0} f\left(\overline{I_\alpha}\right) \cup \bigcup_{\alpha \in \Lambda_0} adh_\beta(V_\alpha) \cup \bigcup_{\alpha \in \Lambda_0} f(J_\alpha)$. Consecuently $Y \backslash \bigcup_{\alpha \in \Lambda_0} adh_\beta(V_\alpha) \in f\left(\overline{\mathcal{I}}\right)$.

Now we present some properties of the $\mathcal{D}$-continuous functions, in the case in which the codomain is a Hausdorf space.

Theorem 7.3
(1) If $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ and $g : (X, \tau, \mathcal{I}) \to (Y, \beta)$ are $\mathcal{D}$-continuous functions, and if (Y, β) is T_2, then $A = \{x \in X : f(x) = g(x)\}$ is closed in $(X, \tau \oplus \mathcal{I})$.
(2) If $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is $\mathcal{D}$-continuous and (Y, β) is T_2, then $A = \{(u, v) \in X \times X : f(u) = f(v)\}$ is closed in $(X \times X, (\tau \oplus \mathcal{I}) \times (\tau \oplus \mathcal{I}))$.

Proof.
(1) Suppose that $u \in X \backslash A$. There is a $\{U, V\} \subseteq \beta$ with $f(u) \in U$, $g(u) \in V$ and $U \cap V = \emptyset$. Then $u \in f^{-1}(U) \cap g^{-1}(V)$, $A \cap [f^{-1}(U) \cap g^{-1}(V)] = \emptyset$ and $f^{-1}(U) \cap g^{-1}(V) \in \tau \oplus \mathcal{I}$, because $f^{-1}(U) \cap g^{-1}(V)$ is open-$\mathcal{I}$.
(2) Suppose that $(u, v) \in (X \times X) \backslash A$. There is a $\{U, V\} \subseteq \beta$ with $f(u) \in U$, $f(v) \in V$ and $U \cap V = \emptyset$. Then $(u, v) \in f^{-1}(U) \times f^{-1}(V)$, $A \cap [f^{-1}(U) \times g^{-1}(V)] = \emptyset$ and $f^{-1}(U) \times f^{-1}(V) \in (\tau \oplus \mathcal{I}) \times (\tau \oplus \mathcal{I})$, because $f^{-1}(U)$ and $f^{-1}(V)$ are open-$\mathcal{I}$. This allows us to conclude that A is closed in the space $(X \times X, (\tau \oplus \mathcal{I}) \times (\tau \oplus \mathcal{I}))$.

Recall that an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{J}$-Hausdorff [13] if for

each $\{a, b\} \subseteq X$ with $a = b$, there exists $\{U, V\} \subseteq \tau$ such that $a \in U$, $b \in V$ and $U \cap V \in \mathcal{I}$.

Theorem 7.4 If $f : (X, \tau, \mathcal{I}) \to (Y, \beta)$ is biyective and $\mathcal{D}$ -open, and if $(X, \tau, \mathcal{I})$ is $\mathcal{J}$ -Hausdorff, then $(Y, \beta \oplus f(\mathcal{I}), f(\mathcal{I}))$ is $\mathcal{J}$ -Hausdorff.

Proof. Suppose that $\{a, b\} \subseteq X$ and that $f(a) = f(b)$. There exists $\{U, V\} \subseteq \tau$ such that $a \in U$, $b \in V$ and $U \cap V \in \mathcal{I}$. Given that $int(f(U)) \cap int(f(V)) \subseteq f(U) \cap f(V) = f(U \cap V) \in f(\mathcal{I})$, we have that $int(f(U)) \cap int(f(V)) \in f(\mathcal{I})$. On the other hand, there exists $\{I_1, I_2\} \subseteq \mathcal{I}$ such that $f(U \backslash I_1) \subseteq int(f(U))$ and $f(V \backslash I_2) \subseteq int(f(V))$.

Hence:

(i) $f(a) \in int(f(U)) \cup f(I_1)$, $f(b) \in int(f(V)) \cup f(I_2)$,

(ii) $\{int(f(U)) \cup f(I_1), int(f(V)) \cup f(I_2)\} \subseteq \beta \oplus f(\mathcal{I})$ and (iii) $[int(f(U)) \cup f(I_1)] \cap [int(f(V)) \cup f(I_2)] = [int(f(U)) \cap int(f(V))] \cup [f(I_1) \cap int(f(V))] \cup [int(f(U)) \cap f(I_2)] \cup [f(I_1) \cap f(I_2)] \in f(\mathcal{I})$.

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