International Journal of Pure and Applied Mathematics
Volume 114 No. 2 2017, 277-292
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v114i2.10

ijpam.eu

TWO NEW FORMS OF QUASI-H-CLOSEDNESS VIA IDEALS

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Abstract: In this paper we introduce two new ideal topological spaces, which are strong forms of the QHC spaces and the $\rho\mathcal{I}$ -QHC spaces. We present several properties and characterizatios of these new spaces.

AMS Subject Classification: 54D30, 54C10
Key Words: C-compact , quasi-H-closed, $\mathcal{I}$-compact, C($\mathcal{I}$)-compact, $\mathcal{I}$-QHC, $\rho\mathcal{I}$-compact, ρC($\mathcal{I}$)-compact, $\rho\mathcal{I}$-QHC, $\sigma\mathcal{I}$-compact

1. Introduction and Preliminaries

The author has introduced the $\rho\mathcal{I}$-compact and the $\sigma\mathcal{I}$-compact spaces, as well as the $\rho\mathcal{I}$-QHC and the ρC($\mathcal{I}$)-compact spaces. With the same spirit, in this article we define and study two new ideal topological spaces, and we establish relationships between these spaces and those initially mentioned, as well as the compact spaces, C-compact spaces and QHC spaces.

An ideal $\mathcal{I}$ in a set X is a subset of $\mathcal{P}(X)$, the power set of X, such that: (i) if $A \subseteq B \subseteq X$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$, and (ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.

Some useful ideals in X are: (i) $\mathcal{P}(A)$, where $A \subseteq X$, (ii) $\mathcal{I}_f\,(X)$, the ideal of all finite subsets of X, (iii) $\mathcal{I}_c\,(X)$, the ideal of all countable subsets of X, (iv) $\mathcal{I}_n\,(X,\tau)$, the ideal of all nowhere dense subsets in a topological space (X,τ).

Received: December 15, 2016
Revised: April 7, 2017
Published: May 6, 2017

If (X,τ) is a topological space and $\mathcal{I}$ is an ideal in X, then $(X,\tau,\mathcal{I})$ is called an *ideal space.*

If $(X,\tau,\mathcal{I})$ is an ideal space then the set $\mathcal{B} = \{U \backslash I : U \in \tau \text{ and } I \in \mathcal{I}\}$ is a base for a topology τ^*, finer than τ.

If (X,τ) is a topological space and $A \subseteq X$ then the closure and the interior of A are denoted by $\overline{A}$ (or $adh\,(A)$, or $adh_\tau\,(A)$) and $\overset{0}{A}$ (or $int(A)$, or $int_\tau(A)$), respectively.

If (X,τ) is a space and $A \subseteq X$ then A is said to be: (1) *regular open* if $A = \overset{0}{\overline{A}}$, (2) *regular closed* if $A = \overline{\overset{0}{A}}$, (3) *pre-open* [3] if $A \subseteq \overset{0}{\overline{A}}$, (4) *$\alpha$-open* [5] if $A \subseteq \overset{0}{\overline{\overset{0}{A}}}$. The set of all preopen subsets of X is denoted by $PO(X)$. Observe that open $\Rightarrow$ α-open $\Rightarrow$ pre-open.

Moreover, if $\mathcal{I}$ is an ideal in X and $\mathcal{I} \cap \tau = \{\varnothing\}$, $\mathcal{I}$ is called *codense* [1]. If $\mathcal{I} \cap PO(X) = \{\varnothing\}$ then $\mathcal{I}$ is said to be *completely codense* [1].

A topological space (X,τ) is said to be: (1) *quasi-H-closed*, or simply *QHC* [8], if for each open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X = \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

(2) *C-compact* [10] if for each $F \subseteq X$, closed, and each open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of F, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

An ideal space $(X,\tau,\mathcal{I})$ is defined to be:

(1) *$\mathcal{I}$-compact* [4] if for all open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $X \backslash \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{I}$.

(2) *$\mathcal{I}$-QHC* [2] if for all open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $X \backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

(3) *$\rho\mathcal{I}$-compact* [6] if for each family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X, if $X \backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X \backslash \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{I}$.

(4) *$\sigma\mathcal{I}$-compact* [6] if for each nonempty family $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty open subsets of X, if $X \backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$.

2. $\sigma\mathcal{I}$-QHC Spaces

The $\mathcal{I}$-QHC spaces are due to Gupta-Noiri [2], and these are generalizations of the QHC spaces of Porter-Thomas [8]. In this section we define the $\sigma\mathcal{I}$-QHC spaces, which are other strong form of $\mathcal{I}$-QHC spaces. We study some of its properties and characterizations.

We recall that if $(X,\tau,\mathcal{I})$ is an ideal space and $A\subseteq X$, then A is said to be *$\rho\mathcal{I}$-QHC* [7] if for all family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X, if $A\setminus\bigcup\limits_{\alpha\in\Lambda}V_\alpha\in\mathcal{I}$, there exists $\Lambda_0\subseteq\Lambda$, finite, with $A\setminus\bigcup\limits_{\alpha\in\Lambda_0}\overline{V_\alpha}\in\mathcal{I}$. The ideal space $(X,\tau,\mathcal{I})$ is said to be *$\rho\mathcal{I}$-QHC* if X is $\rho\mathcal{I}$-QHC.

Definition 2.1 If $(X,\tau,\mathcal{I})$ is an ideal space and $A\subseteq X$, then A is said to be *$\sigma\mathcal{I}$-QHC* if for all nonempty family $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty open subsets of X, if $A\setminus\bigcup\limits_{\alpha\in\Lambda}V_\alpha\in\mathcal{I}$, there exists $\Lambda_0\subseteq\Lambda$, finite, with $A\subseteq\bigcup\limits_{\alpha\in\Lambda_0}\overline{V_\alpha}$. The ideal space $(X,\tau,\mathcal{I})$ is said to be *$\sigma\mathcal{I}$-QHC* if X is $\sigma\mathcal{I}$-QHC.

Note that if $(X,\tau,\mathcal{I})$ is $\sigma\mathcal{I}$-QHC and λ is a topology in X, with $\lambda\subseteq\tau$, then $(X,\lambda,\mathcal{I})$ is $\sigma\mathcal{I}$-QHC.

Moreover it is clear that $(X,\tau,\{\varnothing\})$ is $\sigma\{\varnothing\}$-QHC$\Leftrightarrow(X,\tau,\{\varnothing\})$ is $\{\varnothing\}$-QHC$\Leftrightarrow(X,\tau,\{\varnothing\})$ is $\rho\{\varnothing\}$-QHC$\Leftrightarrow(X,\tau)$ is QHC.

It is also evident that $\sigma\mathcal{I}$-QHC $\Rightarrow$ $\rho\mathcal{I}$-QHC and $\sigma\mathcal{I}$-QHC $\Rightarrow$ QHC. The converse, in general, are not true.

Example 2.1 (1) If $\mathcal{U}$ is the usual topology on $\mathbb{R}$ and if $\mathcal{I}=\mathcal{P}(\mathbb{R})$, then the space $(\mathbb{R},\mathcal{U},\mathcal{I})$ is $\rho\mathcal{I}$-QHC, but $(\mathbb{R},\mathcal{U},\mathcal{I})$ is not $\sigma\mathcal{I}$-QHC.

This implies that, in general, $\rho\mathcal{I}$-QHC $\nRightarrow$ $\sigma\mathcal{I}$-QHC.

(2) We denote by $2\mathbb{Z}$ the set of even integers, and by $2\mathbb{Z}+1$ the set of odd integers. Let τ the topology on $\mathbb{Z}$ given by: If $V\subseteq\mathbb{Z}$ then $V\in\tau\Leftrightarrow$ [if $0\in V$ then $2\mathbb{Z}\subseteq V$, and if $1\in V$ then $2\mathbb{Z}+1\subseteq V$]. Let $\mathcal{I}=\mathcal{P}[(2\mathbb{Z}+1)\cup\{0\}]$. We know that $(\mathbb{Z},\tau)$ is a QHC space and that $(\mathbb{Z},\tau,\mathcal{I})$ is not $\rho\mathcal{I}$-QHC [7]. Hence $(\mathbb{Z},\tau,\mathcal{I})$ is not $\sigma\mathcal{I}$-QHC.

(3) Suppose that $X=\mathbb{R}^+$, $\tau=\{\varnothing,X,X\setminus\mathbb{Z}^+\}\cup\{V_n:n\in\mathbb{Z}^+\}$, where $V_n=\bigcup\limits_{k=0}^{n-1}(k,k+1)$, and $\mathcal{I}=\{A:A\subseteq(0,1)\}$. Then the ideal space $(X,\tau,\mathcal{I})$ is $\sigma\mathcal{I}$-QHC. In fact, if $\{W_\alpha\}_{\alpha\in\Lambda}$ is a nonempty family of nonempty open subsets of X with $X\setminus\bigcup\limits_{\alpha\in\Lambda}W_\alpha\in\mathcal{I}$, then there exists $\alpha_0\in\Lambda$ such that $1\in W_{\alpha_0}$, and so $W_{\alpha_0}=X$.

We can easily see that each closed and open subset of a $\sigma\mathcal{I}$-QHC space is $\sigma\mathcal{I}$-QHC.

Now we present some characterizations of $\sigma\mathcal{I}$-QHC spaces. The proof is similar to Theorems 3.2 and 3.3, so we omit it.

Theorem 2.1 *For an ideal space $(X,\tau,\mathcal{I})$, the following statements are equivalents:*

(1) $(X,\tau,\mathcal{I})$ is $\sigma\mathcal{I}$-QHC.

(2) For each nonempty family $\{F_\alpha\}_{\alpha\in\Lambda}$ of closed subsets of X, with $F_\alpha \neq X$ for each $\alpha \in \Lambda$, if $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $\bigcap\limits_{\alpha\in\Lambda_0} \overset{0}{F_\alpha} = \varnothing$.

(3) For each nonempty family $\{F_\alpha\}_{\alpha\in\Lambda}$ of closed subsets of X, with $F_\alpha \neq X$ for each $\alpha \in \Lambda$, if $\left\{\overset{0}{F_\alpha} : \alpha \in \Lambda\right\}$ has the finite intersection property, then $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \notin \mathcal{I}$.

(4) For each nonempty family $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty regular open subsets of X, if $X\setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, there is $\Lambda_0 \subseteq \Lambda$, finite, such that $X = \bigcup\limits_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

(5) For each nonempty family $\{F_\alpha\}_{\alpha\in\Lambda}$ of regular closed subsets of X, with $F_\alpha \neq X$ for each $\alpha \in \Lambda$, if $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \in \mathcal{I}$, there is $\Lambda_0 \subseteq \Lambda$, finite, such that $\bigcap\limits_{\alpha\in\Lambda_0} \overset{0}{F_\alpha} = \varnothing$.

(6) For each nonempty family $\{F_\alpha\}_{\alpha\in\Lambda}$ of regular closed sets, with $F_\alpha \neq X$ for each $\alpha \in \Lambda$, if $\left\{\overset{0}{F_\alpha} : \alpha \in \Lambda\right\}$ has the finite-intersection property, then $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \notin \mathcal{I}$.

(7) For each open filter base Ω on X, if $\Omega \subseteq \mathcal{P}(X)\setminus\{\varnothing\}$ one has $\bigcap\limits_{V\in\Omega} \overline{V} \notin \mathcal{I}$.

A result in [9] implies that if (X,τ) is a topological space and $\mathcal{I}$ is a completely codense ideal in X, then (X,τ) and (X,τ^*) have the same regular open subsets, and $adh_\tau(V) = adh_{\tau^*}(V)$, for all $V \in \tau^*$. Then the following result is clear.

Theorem 2.2 *If $\mathcal{I}$ is a completely codense ideal in X, the space $(X,\tau,\mathcal{I})$ is $\sigma\mathcal{I}$-QHC if and only if $(X,\tau^*,\mathcal{I})$ is $\sigma\mathcal{I}$-QHC.*

In the Theorem 2.4 we review the behavior of $\sigma\mathcal{I}$-QHC spaces under continuous or open functions. In his proof we use the following lemma, which is due to Newcomb [4].

Lema 2.3 [Newcomb] *Suppose that (X,τ) and (Y,β) are topological spaces and that $f : X \to Y$ is a function. Then:*

(1) *If* $\mathcal{I}$ *is an ideal in* X*, we have that* $f(\mathcal{I}) = \{f(I) : I \in \mathcal{I}\}$ *is an ideal in* Y*.*

(2) *If* f *is inyective and* $\mathcal{J}$ *is an ideal in* Y*, we have that the set* $f^{-1}(\mathcal{J}) = \left\{f^{-1}(J) : J \in \mathcal{J}\right\}$ *is an ideal on* X*.*

Theorem 2.4 *(1) If* $(X, \tau, \mathcal{I})$ *is* $\sigma\mathcal{I}$*-QHC and* $f : (X, \tau) \to (Y, \beta)$ *is a biyective continuous function, then* $(Y, \beta, f(\mathcal{I}))$ *is* $\sigma f(\mathcal{I})$*-QHC.*

(2) If $(X, \tau, \mathcal{I})$ *is* $\sigma\mathcal{I}$*-QHC,* $f : (X, \tau) \to (Y, \beta)$ *is a sobreyective and continuous function and if* $\mathcal{J} = \left\{V \subseteq Y : f^{-1}(V) \in \mathcal{I}\right\}$*, then* $(Y, \beta, \mathcal{J})$ *is* $\sigma\mathcal{J}$*-QHC.*

(3) If $(Y, \beta, \mathcal{J})$ *is* $\sigma\mathcal{J}$*-QHC and* $f : (X, \tau) \to (Y, \beta)$ *is a biyective and open function, then* $\left(X, \tau, f^{-1}(\mathcal{J})\right)$ *is* $\sigma f^{-1}(\mathcal{J})$*-QHC.*

(4) If $(X, \tau, \mathcal{I})$ *is* $\sigma\mathcal{I}$*-QHC,* $f : (X, \tau) \to (Y, \beta)$ *is a sobreyective and continuous function, and if* $\mathcal{J}$ *is an ideal on* Y *with* $\left\{f^{-1}(J) : J \in \mathcal{J}\right\} \subseteq \mathcal{I}$*, then* $(Y, \beta, \mathcal{J})$ *is* $\sigma\mathcal{J}$*-QHC.*

(5) If $(Y, \beta, \mathcal{J})$ *is* $\sigma\mathcal{J}$*-QHC,* $f : (X, \tau) \to (Y, \beta)$ *is a biyective and open function, and if* $\mathcal{I}$ *is an ideal on* X *with* $f(\mathcal{I}) \subseteq \mathcal{J}$*, then* $(X, \tau, \mathcal{I})$ *is* $\sigma\mathcal{I}$*-QHC.*

Proof. (1) Suppose that $\{W_\alpha\}_{\alpha\in\Lambda}$ is a nonempty family of nonempty open subsets of Y with $Y \setminus \bigcup\limits_{\alpha\in\Lambda} W_\alpha \in f(\mathcal{I})$. There exists $I \in \mathcal{I}$ such that $Y \setminus \bigcup\limits_{\alpha\in\Lambda} W_\alpha = f(I)$. Since $X \setminus \bigcup\limits_{\alpha\in\Lambda} f^{-1}(W_\alpha) = f^{-1}(f(I)) = I \in \mathcal{I}$ and $(X, \tau, \mathcal{I})$ is $\sigma\mathcal{I}$-QHC, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X = \bigcup\limits_{\alpha\in\Lambda_0} \overline{f^{-1}(W_\alpha)}$. Given that f is sobreyective and continuous, $Y = \bigcup\limits_{\alpha\in\Lambda_0} f\left(\overline{f^{-1}(W_\alpha)}\right) \subseteq \bigcup\limits_{\alpha\in\Lambda_0} \overline{f\left(f^{-1}(W_\alpha)\right)} = \bigcup\limits_{\alpha\in\Lambda_0} \overline{W_\alpha}$, and so $Y = \bigcup\limits_{\alpha\in\Lambda_0} \overline{W_\alpha}$.

(2) It is simple to see that $\mathcal{J}$ is an ideal in Y. Suppose that $\{W_\alpha\}_{\alpha\in\Lambda}$ is a nonempty family of nonempty open subsets of Y with $Y \setminus \bigcup\limits_{\alpha\in\Lambda} W_\alpha \in \mathcal{J}$. Since $X \setminus \bigcup\limits_{\alpha\in\Lambda} f^{-1}(W_\alpha) = f^{-1}\left(Y \setminus \bigcup\limits_{\alpha\in\Lambda} W_\alpha\right) \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X = \bigcup\limits_{\alpha\in\Lambda_0} \overline{f^{-1}(W_\alpha)}$. Given that f is sobreyective and continuous, $Y = \bigcup\limits_{\alpha\in\Lambda_0} f\left(\overline{f^{-1}(W_\alpha)}\right) \subseteq \bigcup\limits_{\alpha\in\Lambda_0} \overline{f\left(f^{-1}(W_\alpha)\right)} = \bigcup\limits_{\alpha\in\Lambda_0} \overline{W_\alpha}$, and so $Y = \bigcup\limits_{\alpha\in\Lambda_0} \overline{W_\alpha}$.

(3) Suppose that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a nonempty family of nonempty open subsets of X, with $X \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in f^{-1}(\mathcal{J})$. There exists $J \in \mathcal{J}$ such that $X \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha = f^{-1}(J)$. Then $Y \setminus \bigcup\limits_{\alpha\in\Lambda} f(V_\alpha) = J$, and given that $(Y, \beta, \mathcal{J})$ is $\sigma\mathcal{J}$-QHC, there

is $\Lambda_0 \subseteq \Lambda$, finite, with $Y = \bigcup_{\alpha\in\Lambda_0} \overline{f(V_\alpha)}$. Given that f is open and biyective, f is closed, and this implies $\overline{f(V_\alpha)} \subseteq f\left(\overline{V_\alpha}\right)$, for each $\alpha \in \Lambda_0$. Hence $Y = \bigcup_{\alpha\in\Lambda_0} f(\overline{V_\alpha}) = f\left(\bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}\right)$, and then $X = \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

(4) It is similar to (2).

(5) Since f is inyective, $\mathcal{I} = f^{-1}(f(\mathcal{I}))$. Moreover, if $(Y,\beta,\mathcal{J})$ is $\sigma\mathcal{J}$-QHC then $(Y,\beta,f(\mathcal{I}))$ is $\sigma f(\mathcal{I})$-QHC. It is enough now to apply (3).

We end this section by presenting a characterization of $\sigma\mathcal{I}$-QHC spaces, in terms of pre-open and α-open subsets. The proof is similar to that of Theorem 3.7.

Theorem 2.5 *If $(X,\tau,\mathcal{I})$ is an ideal space, the following statements are equivalents:*

(1) $(X,\tau,\mathcal{I})$ is $\sigma\mathcal{I}$-QHC.

(2) For each nonempty family $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty pre-open subsets of X, if $X \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ then there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X = \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

(3) For each nonempty family $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty α-open subsets of X, if $X \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ then there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X = \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

3. σC($\mathcal{I}$)-Compact Spaces

The concept of C($\mathcal{I}$)-compactness is due to Gupta-Noiri, and this is a generalization of C-compactness of Viglino. In this section we introduce and study the $\sigma C(\mathcal{I})$-compact spaces, which are other strong form of C($\mathcal{I}$)-compactness. We present some of its properties and characterizations.

An ideal space $(X,\tau,\mathcal{I})$ is said to be: (1) *C($\mathcal{I}$)-compact* [2] if for each $F \subseteq X$, closed, and each open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of F, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F \setminus \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

(2) *ρC($\mathcal{I}$)-compact* [7] if for each closed subset F of X, and each family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X such that $F \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F \setminus \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

Observe that C-compact $\Rightarrow$ QHC, ρC($\mathcal{I}$)compact $\Longrightarrow$ C($\mathcal{I}$)-compact $\Rightarrow$ $\mathcal{I}$-QHC and that if (X,τ) is C-compact then $(X,\tau,\mathcal{I})$ is C($\mathcal{I}$)-compact.

Definition 3.1 The ideal space $(X,\tau,\mathcal{I})$ is defined to be *σC($\mathcal{I}$)-compact* if for each closed subset F of X, and each nonempty family $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty open subsets of X such that $F\backslash \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F \subseteq \bigcup\limits_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

Note that if $(X,\tau,\mathcal{I})$ is σC($\mathcal{I}$)-compact and λ is a topology in X, with $\lambda \subseteq \tau$, then $(X,\lambda,\mathcal{I})$ is σC($\mathcal{I}$)-compact. En particular, if $(X,\tau^*,\mathcal{I})$ is σC($\mathcal{I}$)-compact then $(X,\tau,\mathcal{I})$ is σC($\mathcal{I}$)-compact.

It is also clear that:

(1) (X,τ) is C-compact $\Leftrightarrow$ $(X,\tau,\{\varnothing\})$ is σC($\{\varnothing\}$)-compact $\Leftrightarrow$ $(X,\tau,\{\varnothing\})$ is C($\{\varnothing\}$)-compact$\Leftrightarrow$ $(X,\tau,\{\varnothing\})$ is ρC($\{\varnothing\}$)-compact.

(2) σC($\mathcal{I}$)-compact $\Rightarrow$ $\sigma\mathcal{I}$-QHC.

(3) σC($\mathcal{I}$)-compact $\Rightarrow$ C-compact.

(4) σC($\mathcal{I}$)-compact $\Rightarrow$ ρC($\mathcal{I}$)-compact.

These implications are, in general, irreversible.

Example 3.1 (1) The space $(\mathbb{R},\mathcal{U},\mathcal{I}=\mathcal{P}(\mathbb{R}))$ of the Example 2.1 is ρC($\mathcal{I}$)-compact. However, given that this space is not $\sigma\mathcal{I}$-QHC, we have that $(\mathbb{R},\mathcal{U},\mathcal{I})$ is not σC($\mathcal{I}$)-compact. Hence, in general, ρC($\mathcal{I}$)-compact $\not\Rightarrow$ σC($\mathcal{I}$)-compact.

(2) We consider again the ideal space $(\mathbb{Z},\tau,\mathcal{I})$ of the Example 2.1. We know that (Z,τ) is C-compact but $(\mathbb{Z},\tau,\mathcal{I})$ is not $\rho\mathcal{I}$-QHC [7]. Thus $(\mathbb{Z},\tau,\mathcal{I})$ is not σC($\mathcal{I}$)-compact.

Thus, in general, C-compact$\not\Rightarrow$ σC($\mathcal{I}$)-compact.

(3) Let $\mathcal{U}$ be the usual topology on $X=[0,1]$. Let $F=\{1/n : n\in\mathbb{Z}^+\}$. We consider the topology $\mathcal{U}^*$ on X generated by $\mathcal{U}\cup\{X\backslash F\}$. A base for $\mathcal{U}^*$ is $\mathcal{B}=\mathcal{U}\cup\{V\backslash F : V\in\mathcal{U}\}$. We know that $(X,\mathcal{U}^*)$ is QHC but is not C-compact [7]. Hence $(X,\mathcal{U}^*,\{\varnothing\})$ is $\sigma\,\{\varnothing\}$-QHC but is not σC($\{\varnothing\}$)-compact.

Then, in general, $\sigma\mathcal{I}$-QHC$\not\Rightarrow$ σC($\mathcal{I}$)-compact.

Theorem 3.1 *If the space $(X,\tau,\mathcal{I})$ is $\sigma\mathcal{I}$-compact then $(X,\tau,\mathcal{I})$ is σC($\mathcal{I}$)-compact.*

Proof. Suppose that K is a closed subset of X, and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a nonempty family of nonempty open subsets of X with $K\backslash \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, this is, $X\backslash \left[(X\backslash K)\cup \bigcup\limits_{\alpha\in\Lambda} V_\alpha\right] \in \mathcal{I}$. By hypothesis, there exists $\Lambda_0\subseteq\Lambda$, finite, such that $X=(X\backslash K)\cup\bigcup\limits_{\alpha\in\Lambda_0} V_\alpha$, and so $K\subseteq \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha \subseteq \bigcup\limits_{\alpha\in\Lambda_0}\overline{V_\alpha}$.

The converse of this theorem, in general, is not true as we can see in the following example.

Example 3.2 If $\mathcal{C} = \{\varnothing, \mathbb{R}\} \cup \{(r, +\infty) : r \in \mathbb{R}\}$, then is easy to see that the space $(\mathbb{R}, \mathcal{C}, \mathcal{I} = \mathcal{P}((-\infty, 0]))$ is σC($\mathcal{I}$)-compact, but is not $\sigma\mathcal{I}$-compact.

In consecuense, we have the next diagram.

$$\begin{tikzcd}
Compact \arrow[r] \arrow[d] & C-Compact \arrow[r] \arrow[d] & QHC \arrow[d] \\
\mathcal{I}-Compact \arrow[r] & C(\mathcal{I})-Compact \arrow[r] & \mathcal{I}-QHC \\
\rho\mathcal{I}-Compact \arrow[r] \arrow[u] & \rho C(\mathcal{I})-Compact \arrow[r] \arrow[u] & \rho\mathcal{I}-QHC \arrow[u] \\
\sigma\mathcal{I}-Compact \arrow[r] \arrow[u] \arrow[d] & \sigma C(\mathcal{I})-Compact \arrow[r] \arrow[u] \arrow[d] & \sigma\mathcal{I}-QHC \arrow[u] \arrow[d] \\
Compact \arrow[r] & C-Compact \arrow[r] & QHC
\end{tikzcd}$$

The first characterizations of σC($\mathcal{I}$)-compactness are presented immediately.

Theorem 3.2 *For an ideal space $(X, \tau, \mathcal{I})$, the following statements are equivalents:*

(1) $(X, \tau, \mathcal{I})$ is σC($\mathcal{I}$)-compact.

(2) For each $F \subseteq X$, closed, and each nonempty family $\{F_\alpha\}_{\alpha\in\Lambda}$ of closed sets, with $F_\alpha \neq X$ for each $\alpha \in \Lambda$, if $\bigcap_{\alpha\in\Lambda}(F \cap F_\alpha) \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $\bigcap_{\alpha\in\Lambda_0}(F \cap \overset{0}{F}_\alpha) = \varnothing$.

(3) For all $F \subseteq X$, closed, and each nonempty family $\{F_\alpha\}_{\alpha\in\Lambda}$ of closed subsets of X, with $F_\alpha \neq X$ for each $\alpha \in \Lambda$, if $\left\{F \cap \overset{0}{F}_\alpha : \alpha \in \Lambda\right\}$ has the finite-intersection property, then we have that $\bigcap_{\alpha\in\Lambda}(F \cap F_\alpha) \notin \mathcal{I}$.

(4) For each $F \subseteq X$, closed, and each nonempty family $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty regular open subsets of X, if $F \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, there is $\Lambda_0 \subseteq \Lambda$, finite, such that $F \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

(5) For all $F \subseteq X$, closed, and each nonempty family $\{F_\alpha\}_{\alpha\in\Lambda}$ of regular closed sets, with $F_\alpha \neq X$ for each $\alpha \in \Lambda$, if $\bigcap_{\alpha\in\Lambda}(F \cap F_\alpha) \in \mathcal{I}$, there is $\Lambda_0 \subseteq \Lambda$, finite, such that $\bigcap_{\alpha\in\Lambda_0}(F \cap \overset{0}{F}_\alpha) = \varnothing$.

(6) For each $F \subseteq X$, closed, and each nonempty family $\{F_\alpha\}_{\alpha\in\Lambda}$ of regular closed subsets of X, with $F_\alpha \neq X$ for each $\alpha \in \Lambda$, if $\left\{F \cap \overset{0}{F_\alpha} : \alpha \in \Lambda\right\}$ has the finite-intersection property, then $\bigcap\limits_{\alpha\in\Lambda} (F \cap F_\alpha) \notin \mathcal{I}$.

Proof. The implications (1)⇒(2), (2)⇒(3), (5)⇒(6) are easy to be established.

(3)⇒(4) Let F a closed subset of X and $\{V_\alpha\}_{\alpha\in\Lambda}$ a nonempty family of nonempty regular open subsets of X with $F \backslash \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, or equivalently, $\bigcap\limits_{\alpha\in\Lambda} (F \cap (X \backslash V_\alpha)) \in \mathcal{I}$.

Then, by hypothesis, the family $\{F \cap int\,(X\backslash V_\alpha) : \alpha \in \Lambda\}$ has no the finite-intersection property, and so there exists $\Lambda_0 \subseteq \Lambda$, finite, with $\bigcap\limits_{\alpha\in\Lambda_0} (F \cap int(X\backslash V_\alpha)) = \varnothing$, this is, $F \subseteq \bigcup\limits_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

(4)⇒(5) It is sufficient to note that the complement of a regular closed subset of X is regular open.

(6)⇒(1) Let F a closed subset of X and $\{V_\alpha\}_{\alpha\in\Lambda}$ a nonempty family of nonempty open subsets of X with $F \backslash \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, that is, $\bigcap\limits_{\alpha\in\Lambda} (F \cap (X\backslash V_\alpha)) \in \mathcal{I}$. Since $\overline{int\,(X\backslash V_\alpha)} \subseteq X \backslash V_\alpha$, for all $\alpha \in \Lambda$, we have that $\bigcap\limits_{\alpha\in\Lambda} \left(F \cap \overline{int(X\backslash V_\alpha)}\right) \in \mathcal{I}$.

But $\overline{int\,(X\backslash V_\alpha)}$ is regular closed, for all $\alpha \in \Lambda$. By hypothesis, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $\bigcap\limits_{\alpha\in\Lambda_0} \left(F \cap int\left(\overline{int(X\backslash V_\alpha)}\right)\right) = \varnothing$, and so $\bigcap\limits_{\alpha\in\Lambda_0} (F \cap int(X\backslash V_\alpha)) = \varnothing$, that is, $F \subseteq \bigcup\limits_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

Here we have other characterization of σC($\mathcal{I}$)-compactness using open filter bases.

Theorem 3.3 *The ideal space $(X, \tau, \mathcal{I})$ is $\sigma C(\mathcal{I})$-compact if and only if, for each $F \subseteq X$, closed, and each open filter base Ω on X, if $\{F \cap V : V \in \Omega\} \subseteq \mathcal{P}(X) \backslash \{\varnothing\}$ then one has $\bigcap\limits_{V\in\Omega} \overline{V} \cap F \notin \mathcal{I}$.*

Proof. (⇒) Suppose that $(X, \tau, \mathcal{I})$ is σC($\mathcal{I}$)-compact and that there are $F \subseteq X$, closed, and an open filter base Ω on X such that $V \cap F \neq \varnothing$, for each $V \in \Omega$, and $\bigcap\limits_{V\in\Omega} \overline{V} \cap F \in \mathcal{I}$.

Since $F\backslash \bigcup\limits_{V\in\Omega}\left(X\backslash\overline{V}\right)\in\mathcal{I}$, there is $\{V_1,V_2,...,V_n\}\subseteq\Omega$ with $F\subseteq\bigcup\limits_{i=1}^{n}\overline{X\backslash\overline{V_i}}$ $=\bigcup\limits_{i=1}^{n}\left(X\backslash\overset{0}{\overline{V_i}}\right)\subseteq\bigcup\limits_{i=1}^{n}(X\backslash V_i)=X\backslash\bigcap\limits_{i=1}^{n}V_i$. But there exists $W\in\Omega$ with $W\subseteq\bigcap\limits_{i=1}^{n}V_i$, and so $F\subseteq X\backslash W$, this is, $F\cap W=\varnothing$, absurd.

($\Leftarrow$) Suppose that $(X,\tau,\mathcal{I})$ is not σC($\mathcal{I}$)-compact. There exist $F\subseteq X$, closed, and a nonempty family $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty open subsets of X, such that $F\backslash\bigcup\limits_{\alpha\in\Lambda}V_\alpha\in\mathcal{I}$, but $F\nsubseteq\bigcup\limits_{\alpha\in\Lambda_0}\overline{V_\alpha}$, for each $\Lambda_0\subseteq\Lambda$, finite. In particular, for all $\alpha\in\Lambda$, $F\nsubseteq\overline{V_\alpha}$. We may assume that $\{V_\alpha\}_{\alpha\in\Lambda}$ is closed for finite unions, because otherwise we can replace $\{V_\alpha\}_{\alpha\in\Lambda}$ by the family of all finite unions of elements in $\{V_\alpha\}_{\alpha\in\Lambda}$.

Then the set $\mathcal{B}=\left\{X\backslash\overline{V_\alpha}:\alpha\in\Lambda\right\}$ is an open filter base on X, and $F\cap\left(X\backslash\overline{V_\alpha}\right)\neq\varnothing$, for each $\alpha\in\Lambda$. The hypothesis implies that $\bigcap\limits_{B\in\mathcal{B}}\left(\overline{B}\cap F\right)\notin\mathcal{I}$, this is, $\bigcap\limits_{\alpha\in\Lambda}\left[\overline{X\backslash\overline{V_\alpha}}\cap F\right]\notin\mathcal{I}$. But for each $\alpha\in\Lambda$, $\overline{X\backslash\overline{V_\alpha}}=X\backslash\overset{0}{\overline{V_i}}\subseteq X\backslash V_\alpha$, and so $F\backslash\bigcup\limits_{\alpha\in\Lambda}V_\alpha=\bigcap\limits_{\alpha\in\Lambda}\left[(X\backslash V_\alpha)\cap F\right]\notin\mathcal{I}$, contradiction.

In the following theorem we review the behavior of σC($\mathcal{I}$)-compact spaces under continuous or open functions.

Theorem 3.4 *(1) If $(X,\tau,\mathcal{I})$ is a $\sigma C(\mathcal{I})$-compact space and if $f:(X,\tau)\to(Y,\beta)$ is a continuous and biyective function, then $(Y,\beta,f(\mathcal{I}))$ is $\sigma C(f((\mathcal{I}))$-compact.*

(2) If $(X,\tau,\mathcal{I})$ is a $\sigma C(\mathcal{I})$-compact ideal space, $f:(X,\tau)\to(Y,\beta)$ is a continuous and sobreyective function and if $\mathcal{J}=\left\{V\subseteq Y:f^{-1}(V)\in\mathcal{I}\right\}$, then $(Y,\beta,\mathcal{J})$ is $\sigma C(\mathcal{J})$-compact.

(3) If $(Y,\beta,\mathcal{J})$ is $\sigma C(\mathcal{J})$-compact and if $f:(X,\tau)\to(Y,\beta)$ is an open and biyective function, then $\left(X,\tau,f^{-1}(\mathcal{J})\right)$ is $\sigma C(f^{-1}(\mathcal{J}))$-compact.

(4) If $(X,\tau,\mathcal{I})$ is $\sigma C(\mathcal{I})$-compact, $f:(X,\tau)\to(Y,\beta)$ is a sobreyective and continuous function, and if $\mathcal{J}$ is an ideal on Y with $\left\{f^{-1}(J):J\in\mathcal{J}\right\}\subseteq\mathcal{I}$, then $(Y,\beta,\mathcal{J})$ is $\sigma C(\mathcal{J})$-compact.

(5) If $(Y,\beta,\mathcal{J})$ is $\sigma C(\mathcal{J})$-compact, $f:(X,\tau)\to(Y,\beta)$ is a biyective and open function, and if $\mathcal{I}$ is an ideal on X with $f(\mathcal{I})\subseteq\mathcal{J}$, then $(X,\tau,\mathcal{I})$ is $\sigma C(\mathcal{I})$-compact.

Proof. (1) Let $B\subseteq Y$, closed, and $\{V_\alpha\}_{\alpha\in\Lambda}$ a nonempty family of nonempty open subsets of Y, with $B\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in f(\mathcal{I})$. There exists $I \in \mathcal{I}$ such that $B\backslash \bigcup_{\alpha\in\Lambda} V_\alpha = f(I)$.

Since $f^{-1}(B) \backslash \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) = f^{-1}(f(I)) = I \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $f^{-1}(B) \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{f^{-1}(V_\alpha)} \subseteq \bigcup_{\alpha\in\Lambda_0} f^{-1}\left(\overline{V_\alpha}\right) = f^{-1}\left(\bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}\right)$. Thus $B \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

(2) It is simple to see that $\mathcal{J}$ is an ideal on Y. Suppose that $B \subseteq Y$ is closed and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a nonempty family of nonempty open subsets of Y, with $B\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{J}$.

Given that $f^{-1}(B) \backslash \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) = f^{-1}\left(B\backslash \bigcup_{\alpha\in\Lambda} V_\alpha\right) \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $f^{-1}(B) \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{f^{-1}(V_\alpha)} \subseteq \bigcup_{\alpha\in\Lambda_0} f^{-1}\left(\overline{V_\alpha}\right) = f^{-1}\left(\bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}\right)$. Hence $B \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$, because f is sobreyective,

(3) Note that since f is biyective and open then f is closed. Suppose that $A \subseteq X$ is closed and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a nonempty family of nonempty open subsets of X, with $A\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in f^{-1}(\mathcal{J})$. There exists $J \in \mathcal{J}$ such that $A\backslash \bigcup_{\alpha\in\Lambda} V_\alpha = f^{-1}(J)$, and so $f(A) \backslash \bigcup_{\alpha\in\Lambda} f(V_\alpha) = f\left(A\backslash \bigcup_{\alpha\in\Lambda} V_\alpha\right) = f\left(f^{-1}(J)\right) = J \in \mathcal{J}$. Since $f(A)$ is closed in Y, there is $\Lambda_0 \subseteq \Lambda$, finite, with $f(A) \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{f(V_\alpha)}$. Given that f is closed, $\overline{f(V_\alpha)} \subseteq f\left(\overline{V_\alpha}\right)$ for all $\alpha \in \Lambda_0$, and so $f(A) \subseteq \bigcup_{\alpha\in\Lambda_0} f\left(\overline{V_\alpha}\right) = f\left(\bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}\right)$. Then $A \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

(4) It is similar to (2).

(5) Since f is inyective, $\mathcal{I} = f^{-1}(f(\mathcal{I}))$. Moreover, if $(Y, \beta, \mathcal{J})$ is σC($\mathcal{J}$)-compact then $(Y, \beta, f(\mathcal{I}))$ is σC($f(\mathcal{I})$)-compact. It is enough now to apply (3).

Definition 3.2 If $(X, \tau, \mathcal{I})$ is an ideal space and $A \subseteq X$, A is said to be *$\sigma C(\mathcal{I})$-compact* if for each $F \subseteq A$, closed in A, and for each nonempty family $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty open subsets of X, if $F\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, there exists $\Lambda_0 \subseteq$

Λ, finite, with $F \subseteq \bigcup\limits_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

Example 3.3 If $\mathcal{C} = \{\varnothing, \mathbb{R}\} \cup \{(r, +\infty) : r \in \mathbb{R}\}$ and if $\mathcal{I}$ is any ideal in $\mathbb{R}$, then is easy to see that, in the space $(\mathbb{R}, C, \mathcal{I})$, all subset A is σC($\mathcal{I}$)-compact. More generally, if (X, τ) is a topological space such that, for each $V \in \tau\backslash \{\varnothing\}$, $\overline{V} = X$, and if $\mathcal{I}$ is any ideal in X, then each $A \subseteq X$ is σC($\mathcal{I}$)-compact in $(X, \tau, \mathcal{I})$.

Theorem 3.5 *1) If $(X, \tau, \mathcal{I})$ is $\sigma C(\mathcal{I})$-compact and $A \subseteq X$ is closed, then A is $\sigma C(\mathcal{I})$-compact.*

2) If $(X, \tau, \mathcal{I})$ is an ideal space and $A_1 \subseteq X$ and $A_2 \subseteq X$ are $\sigma C(\mathcal{I})$-compact then $A_1 \cup A_2$ is $\sigma C(\mathcal{I})$-compact.

Proof. 1) It is clear because if B is closed in A, then B is closed in X.

2) Suppose that B is closed in $A_1 \cup A_2$, and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a nonempty family of nonempty open subsets of X with $B\backslash \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$. There exists $G \subseteq X$, closed, such that $B = (A_1 \cup A_2) \cap G = (A_1 \cap G) \cup (A_2 \cap G)$. Since $A_i \cap G$ is closed in A_i and $(A_i \cap G) \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, for each $i \in \{1, 2\}$, there exists $\Lambda_i \subseteq \Lambda$, finite, with $A_i \cap G \subseteq \bigcup\limits_{\alpha\in\Lambda_i} \overline{V_\alpha}$, for each $i \in \{1, 2\}$.

Thus $A_i \cap G \subseteq \bigcup\limits_{\alpha\in\Lambda_1\cup\Lambda_2} \overline{V_\alpha}$, and so $(A_1 \cup A_2) \cap G \subseteq \bigcup\limits_{\alpha\in\Lambda_1\cup\Lambda_2} \overline{V_\alpha}$, this is, $B \subseteq \bigcup\limits_{\alpha\in\Lambda_1\cup\Lambda_2} \overline{V_\alpha}$.

Now we present an additional characterization of σC($\mathcal{I}$)- compactness.

Definition 3.3 If $(X,\tau,\mathcal{I})$ is an ideal space and $Y \subseteq X$, then Y is *closure $\sigma C(\mathcal{I})$-compact* if for all $K \subseteq Y$, closed in Y, and all nonempty family $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty open subsets of X, if $\overline{K}\backslash \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $K \subseteq \bigcup\limits_{\alpha\in\Lambda_0} adh_{\tau_Y}(V_\alpha \cap Y)$.

Example 3.4 Let $\mathcal{U}$ the usual topology for $X = [0, 1]$, $Y = (0, 1]$ and $K \subseteq Y$, closed in Y.

(i) Suppose that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a $\mathcal{U}$-open cover of $\overline{K}$. Since $\overline{K}$ is compact in X, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $\overline{K} \subseteq \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha$, and so $K \subseteq \bigcup\limits_{\alpha\in\Lambda_0} \left(\overline{V_\alpha} \cap Y\right)$. But, for all $\alpha \in \Lambda_0$, $adh_{\mathcal{U}_Y}\, (V_\alpha \cap Y) = \overline{V_\alpha \cap Y} \cap Y = \overline{V_\alpha} \cap Y$, because Y is open in $(X,\mathcal{U})$. Therefore Y is closure σC($\{\varnothing\}$)-compact.

(*ii*) Y is not σC($\{\varnothing\}$)-compact, because $Y \subseteq \bigcup_{0<r<1} (r,1]$, but if $0 < r_1 < r_2 < \cdots < r_n < 1$ then $Y \nsubseteq \bigcup_{i=1}^{n} \overline{(r_i,1]} = \bigcup_{i=1}^{n} [r_i,1] = [r_1,1]$.

It is simple to see that if A is closure σC($\mathcal{I}$)-compact in the space $(X,\beta,\mathcal{I})$, and τ is a topology on X with $\tau \subseteq \beta$, then A is closure σC($\mathcal{I}$)-compact in the space $(X,\tau,\mathcal{I})$.

Theorem 3.6 *The ideal space* $(X,\tau,\mathcal{I})$ *is* σ*C(*$\mathcal{I}$*)-compact if and only if each* $Y \in \tau$ *is closure* σ*C(*$\mathcal{I}$*)-compact.*

Proof. ($\Rightarrow$) Suppose that $(X,\tau,\mathcal{I})$ is σC($\mathcal{I}$)-compact and that $Y \in \tau$.

Let $K \subseteq Y$, closed in Y, and $\{V_\alpha\}_{\alpha\in\Lambda}$ a nonempty family of nonempty open subsets of X with $\overline{K}\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$. Since $\overline{K}$ is closed in X and $(X,\tau,\mathcal{I})$ is σC($\mathcal{I}$)-compact, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $\overline{K} \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$, and so $K \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$. Given that Y is open in X, $adh_{\tau_Y}(V_\alpha \cap Y) = \overline{V_\alpha} \cap Y$, for all $\alpha \in \Lambda_0$.

But $K \subseteq \bigcup_{\alpha\in\Lambda_0} (\overline{V_\alpha} \cap Y) = \bigcup_{\alpha\in\Lambda_0} adh_{\tau_Y}(V_\alpha \cap Y)$.

($\Leftarrow$) Suppose that F is closed in X, and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a nonempty family of nonempty open subsets of X with $F\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$. Let $\alpha_0 \in \Lambda$. The set $Y = X\backslash\overline{V_{\alpha_0}}$ is open in X and $F \cap Y$ is closed in Y.

Since $\overline{F \cap Y} \subseteq F$ we have that $\overline{F \cap Y}\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$. Now,
$\overline{F \cap Y}\backslash \bigcup_{\alpha\in\Lambda} V_\alpha = \overline{F \cap Y}\backslash \bigcup_{\alpha\in\Lambda\backslash\{\alpha_0\}} V_\alpha$. Thus there exists $\Lambda_0 \subseteq \Lambda\backslash\{\alpha_0\}$, finite, such that $F \cap Y \subseteq \bigcup_{\alpha\in\Lambda_0} adh_{\tau_Y}(V_\alpha \cap Y)$. Given that Y is open, $adh_{\tau_Y}(V_\alpha \cap Y)$ $= \overline{V_\alpha} \cap Y \subseteq \overline{V_\alpha}$, and so $F \cap Y \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$, this is $F\backslash\overline{V_{\alpha_0}} \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$. Hence $F \subseteq \overline{V_{\alpha_0}} \cup \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

Now we present a new characterization of σC($\mathcal{I}$)-compactness, by means of pre-open and α-open subsets.

Theorem 3.7 *If* $(X,\tau,\mathcal{I})$ *is an ideal space, the following statements are equivalents:*

1) $(X,\tau,\mathcal{I})$ *is* σ*C(*$\mathcal{I}$*)-compact.*

2) For each $F\subseteq X$, closed, and each nonempty family $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty pre-open subsets of X, if $F\setminus\bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ then there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F \subseteq \bigcup\limits_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

3) For each $F\subseteq X$, closed, and each nonempty family $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty α-open subsets of X, if $F\setminus\bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ then there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F \subseteq \bigcup\limits_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

Proof. It is sufficient to show that 1)⇒2), since open⇒ α-open⇒pre-open.

1) ⇒ 2) Suppose that $F \subseteq X$ is closed and that $\{V_\alpha\}_{\alpha\in\Lambda}$ is a nonempty family of nonempty pre-open subsets of X, with $F\setminus\bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$. Given that $V_\alpha \subseteq \overset{0}{\overline{V_\alpha}}$, for each $\alpha \in \Lambda$, we have that $F\setminus\bigcup\limits_{\alpha\in\Lambda} \overset{0}{\overline{V_\alpha}} \in \mathcal{I}$, and then there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $F \subseteq \bigcup\limits_{\alpha\in\Lambda_0} \overline{\overset{0}{\overline{V_\alpha}}} \subseteq \bigcup\limits_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

Gupta and Noiri [2] characterized C($\mathcal{I}$)-compactness using a weaker form of filter base convergence. We obtain a similar characterization for σC($\mathcal{I}$)-compactness.

Definition 3.4 If $(X,\tau,\mathcal{I})$ is an ideal space and $\mathcal{B}$ is a filter base in X, then $\mathcal{B}$ is said to be *$\sigma\mathcal{I}$-adherent convergent* if for each $V \in \tau$ such that $\left(\bigcap\limits_{B\in\mathcal{B}} \overline{B}\right)\setminus V \in \mathcal{I}$, there exists $B \in \mathcal{B}$ with $B \subseteq V$.

Theorem 3.8 *An ideal space $(X,\tau,\mathcal{I})$ is $\sigma(\mathcal{I})$-compact if and only if each open filter base $\mathcal{B}$ in X, with $\mathcal{B} \subseteq \mathcal{P}(X)\setminus\{\varnothing\}$, is $\sigma\mathcal{I}$-adherent convergent.*

Proof. (→) Suppose that $V \in \tau$, $\mathcal{B}$ is an open filter base in X and that $\mathcal{B} \subseteq \mathcal{P}(X)\setminus\{\varnothing\}$ and $\left(\bigcap\limits_{B\in\mathcal{B}} \overline{B}\right)\setminus V \in \mathcal{I}$, this is $(X\setminus V)\setminus\bigcup\limits_{B\in\mathcal{B}}\left(X\setminus\overline{B}\right) \in \mathcal{I}$. Since $(X,\tau,\mathcal{I})$ is σC($\mathcal{I}$)-compact there is $\mathcal{B}_0 \subseteq \mathcal{B}$, finite, such that $X\setminus V \subseteq \bigcup\limits_{B\in\mathcal{B}_0} \overline{X\setminus\overline{B}} = \bigcup\limits_{B\in\mathcal{B}_0} X\setminus\overset{0}{\overline{B}} = X\setminus\bigcap\limits_{B\in\mathcal{B}_0}\overset{0}{\overline{B}}$. Then $\bigcap\limits_{B\in\mathcal{B}_0} B \subseteq \bigcap\limits_{B\in\mathcal{B}_0}\overset{0}{\overline{B}} \subseteq V$. But there exists $B_0 \in \mathcal{B}$ with $B_0 \subseteq \bigcap\limits_{B\in\mathcal{B}_0} B$. Hence $B_0 \subseteq V$.

(←) Suppose that $(X,\tau,\mathcal{I})$ is not σC($\mathcal{I}$)-compact. There are $F \subseteq X$, closed, and a nonempty collection $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty open sets such that $F\setminus\bigcup\limits_{\alpha\in\Lambda} V_\alpha$

$\in \mathcal{I}$ but $F \nsubseteq \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$, for each $\Lambda_0 \subseteq \Lambda$, finite. In particular $F \nsubseteq \overline{V_\alpha}$, for each $\alpha \in \Lambda$. We may assume that $\{V_\alpha\}_{\alpha\in\Lambda}$ is closed for finite unions. Then the set $\mathcal{B} = \left\{X\backslash\overline{V_\alpha} : \alpha \in \Lambda\right\}$ is an open filter base in X and $\mathcal{B} \subseteq \mathcal{P}(X)\backslash\{\varnothing\}$.

Now $\left(\bigcap_{B\in\mathcal{B}} \overline{B}\right) \backslash (X\backslash F) = F\backslash \bigcup_{\alpha\in\Lambda} \overset{0}{\overline{V_\alpha}} \subseteq F\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, and this implies that $\left(\bigcap_{B\in\mathcal{B}} \overline{B}\right) \backslash (X\backslash F) \in \mathcal{I}$. The hypothesis implies that there exists $B \in \mathcal{B}$ with $B \subseteq X\backslash F$ or, equivalently, there exists $\alpha_0 \in \Lambda$ such that $X\backslash\overline{V_{\alpha_0}} \subseteq X\backslash F$. Thus $F \subseteq \overline{V_{\alpha_0}}$, contradiction.

Finally, we present a sufficient condition for a σC($\mathcal{I}$)-compact space turns to be maximal σC($\mathcal{I}$)-compact.

Definition 3.5 A σC($\mathcal{I}$)-compact space $(X, \tau, \mathcal{I})$ is said to be *maximal $\sigma C(\mathcal{I})$-compact* if for each topology β on X, if $\tau \subseteq \beta$ and $\tau \neq \beta$ then the space $(X, \beta, \mathcal{I})$ is not σC($\mathcal{I}$)-compact.

Theorem 3.9 *Let $(X, \tau, \mathcal{I})$ be a $\sigma C(\mathcal{I})$-compact space such that, for each $A \subseteq X$, if A is closure $\sigma C(\mathcal{I})$-compact and $X\backslash A$ is $\sigma C(\mathcal{I})$-compact, we have that $A \in \tau$. Then $(X, \tau, \mathcal{I})$ is maximal $\sigma C(\mathcal{I})$-compact.*

Proof. Suppose that $(X, \tau, \mathcal{I})$ is not maximal σC($\mathcal{I}$)-compact. Then there is a topology β on X such that $\tau \subseteq \beta$, $\tau \neq \beta$ and $(X, \beta, \mathcal{I})$ is σC($\mathcal{I}$)-compact. Let $A \in \beta\backslash\tau$. Then A is closure σC($\mathcal{I}$)-compact in the space $(X, \beta, \mathcal{I})$, by Theorem 3.6, and so A is closure σC($\mathcal{I}$)-compact in the space $(X, \tau, \mathcal{I})$. Now, $X\backslash A$ is σC($\mathcal{I}$)-compact in the space $(X, \beta, \mathcal{I})$ since that $X\backslash A$ is closed in (X, β).

If $\{W_\alpha\}_{\alpha\in\Lambda}$ is a nonempty family of nonempty elements in $\tau \subseteq \beta$ such that $(X\backslash A) \backslash \bigcup_{\alpha\in\Lambda} W_\alpha \in \mathcal{I}$, then there is $\Lambda_0 \subseteq \Lambda$, finite, with $X\backslash A \subseteq \bigcup_{\alpha\in\Lambda_0} adh_\beta(W_\alpha)$. Given that $\bigcup_{\alpha\in\Lambda_0} adh_\beta(W_\alpha) \subseteq \bigcup_{\alpha\in\Lambda_0} adh_\tau(W_\alpha)$ we conclude that $X\backslash A$ is σC($\mathcal{I}$)-compact in the space $(X, \tau, \mathcal{I})$. The hypothesis implies that $A \in \tau$, contradiction.

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