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DECOHERENCE-FREE SUBSPACES FOR OPEN QUANTUM RANDOM WALKS ON GRAPHS

J. Agredo

Department of Mathematics
National University of Colombia
Department of Mathematics
Colombian School of Engineering Julio Garavito
Bogotá, COLOMBIA

Abstract: We study decoherence-free subspaces in a type of Quantum Markov Semigroups called continuous-time open quantum random walks on graphs. We measure the temporary changes of quantum correlations using geometric quantum discord with bures distance under some assumptions about the semigroup. In particular, we characterize the decay of correlations to zero, showing that turns out to be closely related with the structure of decoherence-free subspace.

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1. Introduction

Continuous-time open quantum random walks (COQRW) on graphs were introduced using the continuous-time limits of open quantum random walks in [19]. It was shown that the limit processes are represented by Quantum Markov Semigroups (QMSs) $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$, i.e., $\mathcal{T}$ is a weakly*-continuous semigroup of completely positive, identity preserving, normal linear maps on the von Neumann algebra $\mathsf{B}(\mathsf{h})$ of all linear bounded operators on a given complex separable

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Hilbert space h. QMSs are a non-commutative extension of Markov semigroups defined in classical probability, they represent an evolution without memory of a microscopic system in accordance with the laws of quantum physics and fit into the framework of open quantum systems (see [12], [13]). The semigroup $\mathcal{T}$ corresponds to the Heisenberg picture in the sense that given any observable x, $\mathcal{T}_t(x)$ describes its evolution at time t. In this way, given a density matrix ρ, its dynamics (Schrodinger picture or predual semigroup) is given by the semigroup $\mathcal{T}_{*t}(\rho)$, where $tr(\rho\mathcal{T}_t(x)) = tr(\mathcal{T}_{*t}(\rho)x)$.

Several aspects of temporal evolutions described by COQRWs have been investigated. By example in [2], and [19], relationship with non-Markovian generalized Lindblad master equations is studied. In [6] (in a particular class of COQRWs) convergence to a steady state regardless of the initial state when a graph is connected is showed. In this work we are interested in study decoherence-free subspaces in continuous-time open quantum random walks on graphs.

Decoherence occurs when a quantum system interacts with its environment in an irreversible way. Decoherence and noise ([4, 14, 15, 20, 21, 25] and references therein) typically affect quantum features of a state over its time evolution, however it may be possible to find states with a unitary evolution in some "good" portion of a system.

Two main approaches to decoherence of open quantum systems have been proposed in the literature; both are based on quantum Markov semigroups.

Blanchard and Olkiewicz [3], starting from an algebraic setting, defined environment induced decoherence and found many physical models where the system algebra decomposes as the direct sum of two pieces: a subalgebra, called the decoherence-free algebra, where the semigroup acts homomorphically, a Banach subspace where the semigroup action is purely dissipative (see e.g. papers [4, 5] and the references therein) and vanishing as time tends to infinity. The decoherence-free subalgebra was later characterised in [11, 13] as the commutant (or generalised commutant for unbounded operators) of certain families of operators arising form the GKSL (Gorini-Kossakowski-Sudarshan-Lindblad) representation of the generator. In particular, decoherence-free subalgebras of COQRWs were studied in [10].

In the approach to decoherence proposed by Lidar et al. [15, 8] registers of a quantum computer are modeled by a quantum open system on a finite-dimensional Hilbert space h. The time evolution of states is described by a predual semigroup $\mathcal{T}_*$ on the Banach space of trace class operators on h. Given that, for a quantum computer to execute a quantum algorithm efficiently, it needs to evolve by unitary quantum dynamics, in this aprroach, is defined a particular sector where a quantum algorithm is executable: the decoherence-free subspaces.

A subspace $\mathsf{h_f}$ of h is *decoherence-free* if the time evolution of states ω supported in $\mathsf{h_f}$ is given by $\omega \to e^{-\mathrm{it}K}\omega e^{\mathrm{it}K}$ for some self-adjoint operator K on $\mathsf{h_f}$.

Decoherence-free subspaces were identified in [8] (see also [24]) under some physical (somewhat implicit) assumptions, we refer to [15] for an introduction to the theory of decoherence-free subspaces with a lot of examples and applications to protection of quantum information.

Lidar et al. papers, however, are concerned only with *finite-dimensional* systems and focus on explicit physical models. Moreover, his method essentially depends on the choice of an orthonormal basis at the outset. This basis is determined by the spectral analysis of the coefficients of the GKSL generator of the quantum Markov semigroup. As a result, this method cannot be extended to infinite dimensions, or to the case of continuous spectra and unbounded coefficients of the GKSL-generator. In [1] they look at the decoherence-free subspace issue from a mathematical point of view and study the following problem: given a quantum Markov semigroup on the algebra $\mathsf{B}(\mathsf{h})$ with generator represented in a generalised GKSL form, characterising its decoherence-free subspaces for a possibly *infinite dimensional* Hilbert space h. We use this approach by study decoherence-free subspaces and temporary changes of quantum correlations in COQRWs.

The structure of the paper is as follows. Section 2 contains the definition of decoherence-free subspaces for any QMSs and some preliminary remarks. In Section 3 we introduce COQRW on graphs and we study his decoherence-free subspace. The temporary change of quantum correlations using geometric quantum discord with bures distance under some assumptions about the semigroup is presented in Section 4.

2. Decoherence-Free Subspaces

Fron now on, we denote the von Neumann algebra of all bounded operators on the Hilbert space h by $\mathsf{B}(\mathsf{h})$. We recall that an state ω on $\mathsf{B}(\mathsf{h})$ is a positive, trace-one, operator on h, in particular it is a clase trace operator on h. The support *supp*(ω) of ω is the closed subspace of h generated by eigenvectors with strictly positive eigenvalues.

Definition 1. A subspace $\mathsf{h_f}$ of h is called *decoherence-free* (DF) if there exists a self-adjoint operator K on $\mathsf{h_f}$ such that for all state ω with support in

$\mathsf{h_f}$ we have

$$\mathcal{T}_{*t}(\omega) = e^{-\mathrm{i}tK}\omega\, e^{\mathrm{i}tK} \tag{1}$$

for all $t \geq 0$. From now on, we will call that K is *associated with* h_f.

Note that a self-adjoint operator K on $\mathsf{h_f}$ can always be extended to the whole Hilbert space h, therefore DF subspaces could be defined in an equivalent way with a self-adjoint operator K on h leaving the subspace $\mathsf{h_f}$ invariant. In a more precise way, for an unbounded K, this means that $e^{-\mathrm{i}tK}(\mathsf{h_f}) \subseteq \mathsf{h_f}$ for all $t \in \mathbb{R}$. Moreover, note that a DF subspace is necessarily closed with respect to the norm topology of h.

Remark. COQRW on graphs are a particular case of predual semigroup of a QMS, where the generator $\mathcal{L}_*$ of predual semigroup has a well-known representation. In general , any generator of a QMS is described by the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form

$$\mathcal{L}_*(\rho) = \rho G^* + \sum_{s\geq 1} L_s \rho L_s^* + G\rho, \tag{2}$$

for all state ρ on $\mathbf{B}(\mathsf{h})$, where

$$G = -\frac{1}{2}\sum_{s\geq 1} L_s^* L_s - iH,$$

$L_s, H \in \mathbf{B}(\mathsf{h})$ wit H selftadjoint, $(L_s)_{s\geq 1}$ is a finite or infinite sequence and the series $\sum_{s\geq 1} L_s^* L_s$ converge strongly .(See [18], theorem 3.16, pag 271.)

Remark. Recall that the operators $L_\ell, H \in \mathbf{B}(\mathsf{h})$ in a GKSL representation of $\mathcal{L}$ are not unique, we may, for instance, translate each L_ℓ by adding multiples $z_\ell \mathbb{1}$ of the identity operator $\mathbb{1}$, with $\sum_\ell |z_\ell|^2 < \infty$. In this way we obtain another GKSL representation of $\mathcal{L}$ with $L'_\ell = L_\ell + z_\ell \mathbb{1}$ and $H' = H + (2\mathrm{i})^{-1}\sum_{\ell\geq 1}(\bar{z}_\ell L_\ell - z_\ell L_\ell^*)$. We refer to [18] p.272–273 for a detailed discussion on this subject.

Using the structure in the generator of semigroup is possible to give necessary and sufficient conditions to find DFs of an arbitrary QMS. This sentence is true when the coefficients of $\mathcal{L}_*$ are bounded operator and its also true in the not bounded case. We present the result in the bounded case. (General result is founded in [1], proposition 7).

Theorem 2. *A subspace $\mathsf{h_f}$ is a DF subspace with associated self-adjoint operator K, if and only if in any GKSL representation of $\mathcal{L}$ by means of operators L_ℓ, G there exist complex numbers λ_ℓ $(\ell \geq 1)$ and a real number r such that $\sum_{\ell\geq 1} |\lambda_\ell|^2 < \infty$ and*

1. $L_\ell u = \lambda_\ell u$ *for all* $u \in \mathsf{h_f}$ *and* $\ell \geq 1$,
2. $(G + \mathrm{i}K)u = -(\frac{1}{2}\sum_{\ell\geq 1}|\lambda_\ell|^2 + \mathrm{i}r)u$ *for all* $u \in \mathsf{h_f}$.

Proof. Consider the GKSL representation of the generator $\mathcal{L}$ and $\mathsf{h_f}$ is a DF subspace with associated operator K.

By the well-known polarisation identity (1) also holds for $\omega = |u\rangle\langle v|$ with $u, v \in \mathsf{h_f}$. Differentiating we have

$$-\mathrm{i}[K, |u\rangle\langle v|] = G|u\rangle\langle v| + \sum_\ell |L_\ell u\rangle\langle L_\ell v| + |u\rangle\langle v|G^*. \tag{3}$$

If $v = u$, for all $w \in \mathsf{h}$ orthogonal to u we find

$$\sum_\ell |\langle w, L_\ell u\rangle|^2 = 0, \tag{4}$$

it follows that u is an eigenvector of all L_ℓ, i.e. $L_\ell u = \lambda_\ell(u)u$ for $\lambda_\ell(u) \in \mathbb{C}$.

The identity (4) for $w \in \mathsf{h_f}$ also yields

$$\lambda_\ell(u)\langle w, u\rangle = \langle w, p_\mathsf{f} L_\ell p_\mathsf{f} u\rangle = \langle p_\mathsf{f} L_\ell^* p_\mathsf{f} w, u\rangle$$

i.e. $p_\mathsf{f} L_\ell^* p_\mathsf{f} w = 0$ if $\langle w, u\rangle = 0$ and $p_\mathsf{f} L_\ell^* p_\mathsf{f} u = \overline{\lambda_\ell}(u)u$ otherwise, showing that the operator $p_\mathsf{f} L_\ell p_\mathsf{f}$ is normal.

We now prove that eigenvalues $\lambda_\ell(u)$ do not depend on the choice of the vector $u \in \mathsf{h_f}$. Note, first of all, that eigenvectors u, v in $\mathsf{h_f}$ of $p_\mathsf{f} L_\ell p_\mathsf{f}$ with different eigenvalues $\lambda_\ell(u) \neq \lambda_\ell(v)$ are orthogonal since

$$\lambda_\ell(v)\langle v, u\rangle = \langle p_\mathsf{f} L_\ell^* p_\mathsf{f} v, u\rangle = \langle v, p_\mathsf{f} L_\ell p_\mathsf{f} u\rangle = \lambda_\ell(u)\langle v, u\rangle.$$

Since, the Hilbert space h being separable, the spectrum of $p_\mathsf{f} L_\ell p_\mathsf{f}$ is at most countable, hence totally disconnected. The function on the unit sphere of $\mathsf{h_f}$

$$u \to \langle u, L_\ell u\rangle = \lambda_\ell(u)$$

is continuous and so its range must be connected. It follows that the function $u \to \lambda_\ell(u)$ is constant.

Now, rewriting (3) as

$$|(G + \mathrm{i}K)u\rangle\langle v| + |u\rangle\langle (G + \mathrm{i}K)v| + \sum_\ell |\lambda_\ell|^2\ |u\rangle\langle v| = 0, \tag{5}$$

we see that u and v are also eigenvectors for $G+\mathrm{i}K$. The eigenvalues $z(u)$ and $z(v)$ fulfill the identity

$$\left(z(u)+\overline{z(v)}+\sum_{\ell}|\lambda_\ell|^2\right)|u\rangle\langle v|=0, \tag{6}$$

hence $z(u)+\overline{z(v)}+\sum_\ell|\lambda_\ell|^2=0$ for all $u,v\in\mathsf{h_f}$. Taking $u=v$ we see that

$$z(u)=-\mathrm{i}r(u)-\frac{1}{2}\sum_{\ell}|\lambda_\ell|^2$$

for some $r(u)\in\mathbb{R}$. Finally, replacing this in (6), we see that $r(u)$ must be independent of $u\in\mathsf{h_f}$.

Conversely, suppose that 1. and 2. hold, then we compute immediately

$$-\mathrm{i}\,[K,|u\rangle\langle v|\,]=\mathcal{L}_*(|u\rangle\langle v|)$$

for all $u,v\in\mathsf{h_f}$. Since $\mathsf{h_f}$ is K-invariant, replacing u,v by

$$e^{-\mathrm{i}(t-s)K}u,e^{-\mathrm{i}(t-s)K}v\in\mathsf{h_f}$$

the above relationship also holds for $\left|e^{-\mathrm{i}(t-s)K}u\rangle\langle e^{-\mathrm{i}(t-s)K}v\right|$ and we have,

$$\begin{aligned}&\frac{d}{ds}\mathcal{T}_{*s}\left(e^{-\mathrm{i}(t-s)K}|u\rangle\langle v|e^{\mathrm{i}(t-s)K}\right)\\ =&\mathcal{T}_{*s}\left((\mathcal{L}_*+\mathrm{i}[K,\cdot])(e^{-\mathrm{i}(t-s)K}|u\rangle\langle v|e^{\mathrm{i}(t-s)K}\right)\\ =&0.\end{aligned}$$

Therefore

$$\mathcal{T}_{*t}\left(|u\rangle\langle v|\right)=e^{-\mathrm{i}tK}\,|u\rangle\langle v|\,e^{\mathrm{i}tK}$$

and $\mathsf{h_f}$ is decoherence-free. □

Remark. The above result shows that, translating the operators L_ℓ by $-\lambda_\ell$, we find another GKSL representation of $\mathcal{L}$ with $L'_\ell=L_\ell-\lambda_\ell\mathbb{1}$ and $H'=H+(2\mathrm{i})^{-1}\sum_\ell(\bar{z}_\ell L_\ell-z_\ell L^*_\ell)$. In this way, since $\sum_{\ell\geq 1}(L'_\ell)^*L'_\ell$ vanishes on $\mathsf{h_f}$, we find as self-adjoint operator K associated with $\mathsf{h_f}$ the generator of the one-parameter group originating from the action of the semigroup in the new GKSL representation of $\mathcal{L}$.

This theorem provides a recipe for finding DF subspaces. First of all look for common eigenspaces for all the operators L_ℓ, then, translate L_ℓ to $L_\ell-\lambda_\ell\mathbb{1}$

with eigenvalues λ_ℓ finding a new GKSL representation of the generator $\mathcal{L}$. The intersection of common eigenspaces of all the operators L_ℓ is now the common kernel of all the operators $L_\ell - \lambda_\ell \mathbb{1}$. Finally check that the operator G founded in the new GSKL representation of $\mathcal{L}$ leaves the common kernel invariant and is anti self-adjoint on this subspace.

Remark. A K-invariant subspace of a DF subspace is itself a DF subspace, therefore we will be interested in *maximal* DF subspaces.

3. COQRWs: Some Properties and Associated Decoherence-Free Subspaces

In this section we describe the model, some properties and DFs of COQRWs.

Let $\mathcal{G}$ be a graph such that the set of vertexes is given by $\mathcal{V} = \{1, \ldots, d\}$, with $d \geq 2$. To each vertex $j \in \mathcal{V}$ we associate an element e_j in the canonical basis of $\mathbb{C}^d$, moreover, all vertex in $\mathcal{V}$ have degrees of freedom given, in this article, by the simplest space $\mathbb{C}^2$ equipped with an orthonormal basis $(f_j)_{j=1,2}$, therefore the dynamic is describe by operators in $\mathsf{h} = \mathbb{C}^2 \otimes \mathbb{C}^d$ in the following way: for all $j, m \in \mathcal{V}$ such that $j \neq m$, we fix $B_{mj} \in M_2(\mathbb{C})$ standing for the effect of passing from the vertex m to the vertex j. We also define the sum of all the effects when one leaves the vertex m

$$B^*_{mm} B_{mm} := \sum_{j \neq m} B^*_{mj} B_{mj}. \tag{7}$$

Proposition 3.

$$Ker(B_{mm}) = \bigcap_{j \neq m} Ker(B_{mj})$$

for all $m \in \mathcal{V}$.

Proof. $x \in Ker(B_{mm})$ if and only if $B_{mm}x = 0$ if and only if $x \in Ker(B^*_{mm} B_{mm})$ if and only if

$$\sum_{j \neq m} B^*_{mj} B_{mj} x = 0,$$

if and only if $\sum_{j \neq m} \|B_{mj}x\|^2 = \sum_{j \neq m} \langle B_{mj}x, B_{mj}x \rangle = 0$ if and only if $B_{mj}x = 0$ for all $j \neq m$ if and only if $x \in \bigcap_{j \neq m} Ker(B_{mj})$. □

The continuous-time open quantum random walk (COQRW) is the predual semigroup $\mathcal{T}_* = (\mathcal{T}_{*t})_{t\geq 0}$ acting on the set of class–trace operator $\mathcal{I}(\mathsf{h})$ defined in $\mathsf{h} = \mathbb{C}^2 \otimes \mathbb{C}^d$, its Lindblad generator is given by

$$\mathcal{L}_*(\rho) = \rho G^* + \sum_{j,m\in\mathcal{V};j\neq m} L_{mj}\rho L_{mj}^* + G\rho, \tag{8}$$

for all $\rho \in \mathcal{I}(\mathsf{h})$, where

$$L_{mj} = B_{mj} \otimes |e_j\rangle\langle e_m|, \quad j \neq m$$

$$G = -iH - \frac{1}{2}\sum_{j\neq m} L_{mj}^* L_{mj} = -iH - \frac{1}{2}\sum_{m=1}^{k} B_{mm}^* B_{mm} \otimes |e_m\rangle\langle e_m|$$

with $H = H^*$ bounded operator in h.

Remark. Let $B : \mathbb{C}^2 \to Rang(B)$, $P : \mathbb{C}^d \to Rang(P)$ be linear operators then

$$Ker(B\otimes P) = \left(Ker(B)\otimes\mathbb{C}^d\right) \oplus \left(\mathbb{C}^2 \otimes Ker(P)\right) \tag{9}$$

(see theorem 2.19 in [9]).

We denote $\{e_m\}^\perp$ to the orthogonal subespace of subspace generated by e_m.

Proposition 4.

$$\bigcap_{j\in\mathcal{V}}[(Ker(B_{mj})\otimes\mathbb{C}^d) \oplus (\mathbb{C}^2\otimes\{e_m\}^\perp)] =$$
$$\left(\bigcap_{j\in\mathcal{V}} Ker(B_{mj})\otimes\mathbb{C}^d\right) \oplus (\mathbb{C}^2\otimes\{e_m\}^\perp)$$

for all $m \in \mathcal{V}$.

Proof. We take

$$\Delta_1 := \bigcap_{j\in\mathcal{V}}[(Ker(B_{mj})\otimes\mathbb{C}^d) \oplus (\mathbb{C}^2\otimes\{e_m\}^\perp)]$$

and

$$\Delta_2 := \left(\bigcap_{j\in\mathcal{V}} Ker(B_{mj})\otimes\mathbb{C}^d\right) \oplus (\mathbb{C}^2\otimes\{e_m\}^\perp).$$

Given $m \in \mathcal{V}$, its clear that $\Delta_2 \subset (Ker(B_{mj}) \otimes \mathbb{C}^d) \oplus (\mathbb{C}^2 \otimes \{e_m\}^\perp)$ for all $j \in \mathcal{V}$ then $\Delta_2 \subset \Delta_1$.

Moreover, $\Delta_1 \subset (Ker(B_{mj}) \otimes \mathbb{C}^d) \oplus (\mathbb{C}^2 \otimes \{e_m\}^\perp)$ for all $j \in \mathcal{V}$ then $\Delta_1 \subset \Delta_2$. $\square$

A description of decoherence-free subspace for COQRW is obtained by applying Theorem 2, Proposition 3, Proposition 4, and equality (9).

Theorem 5. *Let* h_f *be DF subspace of COQRW and*

$$\mathcal{V}_0 := \{i \in \mathcal{V}; B_{ij} = 0 \quad \forall j \neq i\}$$

then

$$\mathsf{h}_f = \bigcap_{m \in \mathcal{V} - \mathcal{V}_0} [(Ker(B_{mm}) \otimes \mathbb{C}^d) \oplus (\mathbb{C}^2 \otimes \{e_m\}^\perp)]$$

with $K = H$ *associated self-adjoint operator.*

Proof. First note that the only eigenvalue of an operator

$$L_{mj} = B_{mj} \otimes |e_j\rangle\langle e_m|$$

with $m \neq j$ and $B_{mj} \neq 0$ is 0. Indeed, if we suppose that exist $u = \sum_{k,s} u_{ks} f_k \otimes e_s$ eigenvector of L_{mj} with eigenvalue associated $\lambda \neq 0$ then

$$\begin{aligned} \lambda \sum_{k,s} u_{ks} f_k \otimes e_s = \lambda u = L_{mj} u &= \sum_{k,s} u_{ks} B_{mj} f_k \otimes |e_j\rangle\langle e_m | e_s \\ &= \sum_k u_{km} B_{mj} f_k \otimes e_j \end{aligned}$$

so $\lambda \sum_{k,s} u_{ks} \langle f_k \otimes e_s, f_r \otimes e_m \rangle = 0$ for all $m \neq j$, i.e., $\lambda u_{rm} = 0$ for all r and $m \neq j$. Thus $u = \sum_k u_{kj} f_k \otimes e_j$ hence

$$\lambda u = \lambda \sum_k u_{kj} f_k \otimes e_j = L_{mj} u = \sum_k u_{kj} B_{mj} f_k \otimes |e_j\rangle\langle e_m | e_j = 0$$

then $u = 0$ (we suppose $\lambda \neq 0$) and by other hand, if u is eigenvector then $u \neq 0$, this is a contradiction. Therefore $\lambda = 0$.

Second,by (9) note that

$$\bigcap_{\substack{m,j \in \mathcal{V} \\ m \neq j}} Ker(L_{mj}) = \bigcap_{\substack{m \in \mathcal{V} - \mathcal{V}_0 \\ j \in \mathcal{V}}} Ker(L_{mj}) = \bigcap_{\substack{m \in \mathcal{V} - \mathcal{V}_0 \\ j \in \mathcal{V}}} Ker(B_{mj} \otimes |e_j\rangle\langle e_m|)$$

$$= \bigcap_{\substack{m\in\mathcal{V}-\mathcal{V}_0 \\ j\in\mathcal{V}}} (Ker(B_{mj}) \otimes \mathbb{C}^d) \oplus (\mathbb{C}^2 \otimes Ker(|e_j\rangle\langle e_m|)$$

$$= \bigcap_{\substack{m\in\mathcal{V}-\mathcal{V}_0 \\ j\in\mathcal{V}}} [(Ker(B_{mj}) \otimes \mathbb{C}^d) \oplus (\mathbb{C}^2 \otimes \{e_m\}^{\perp})].$$

Finally, we see that if $u \in \bigcap_{\substack{m,j\in\mathcal{V} \\ m\neq j}} Ker(L_{mj})$ then $L^*_{mj}L_{mj}u = 0$ for all $m \neq j$. Taking $K = H$ we otain

$$(G + iK)u = i(-H + K)u = 0.$$

Applying Theorem 2 with $K = H$ we obtain that .

$$\mathsf{h}_f = \bigcap_{\substack{m\in\mathcal{V}-\mathcal{V}_0 \\ j\in\mathcal{V}}} [(Ker(B_{mj}) \otimes \mathbb{C}^d) \oplus (\mathbb{C}^2 \otimes \{e_m\}^{\perp})]$$

Using Propositions 3 and 4 it follows that

$$\begin{aligned}
\mathsf{h}_f &= \bigcap_{\substack{m\in\mathcal{V}-\mathcal{V}_0 \\ j\in\mathcal{V}}} [(Ker(B_{mj}) \otimes \mathbb{C}^d) \oplus (\mathbb{C}^2 \otimes \{e_m\}^{\perp})] \\
&= \bigcap_{m\in\mathcal{V}-\mathcal{V}_0} \bigcap_{j\in\mathcal{V}} [(Ker(B_{mj}) \otimes \mathbb{C}^d) \oplus (\mathbb{C}^2 \otimes \{e_m\}^{\perp})] \\
&= \bigcap_{m\in\mathcal{V}-\mathcal{V}_0} \left[\left(\bigcap_{j\in\mathcal{V}} Ker(B_{mj}) \otimes \mathbb{C}^d \right) \oplus (\mathbb{C}^2 \otimes \{e_m\}^{\perp}) \right] \\
&= \bigcap_{m\in\mathcal{V}-\mathcal{V}_0} (Ker(B_{mm}) \otimes \mathbb{C}^d) \oplus (\mathbb{C}^2 \otimes \{e_m\}^{\perp})
\end{aligned}$$

□

Remark. Theorem 5 establish the sectors, where a quantum algorithm describe by COQRWs on graphs is executable.

Hereinafter, we study some properties of COQRWs when $H = 0$, in this case, we obtain that

$$G = -\frac{1}{2} \sum_{j\neq m} L^*_{mj}L_{mj} = -\frac{1}{2} \sum_{m=1}^{k} B^*_{mm}B_{mm} \otimes |e_m\rangle\langle e_m|.$$

Writing any ρ in $\mathbf{h} = \mathbb{C}^2 \otimes \mathbb{C}^{\mathcal{V}}$ as a block matrix with respect to the canonical basis of $\mathbf{B}(\mathbb{C}^{\mathcal{V}})$ then we obtain

$$\rho = \sum_{p,q \in \mathcal{V}} \widetilde{\rho}_{pq} \otimes |e_q\rangle\langle e_p|, \quad \widetilde{\rho}_{pq} \in M_2(\mathbb{C}). \tag{10}$$

We call *diagonal subspace*, and denote it by $\mathcal{D}$ the subspace generated by the elements $X_{kk} \otimes |e_k\rangle\langle e_k|$ with $X_{kk} \in M_2(\mathbb{C})$. Let $\mathcal{E} : \mathbf{B}(\mathbf{h}) \to \mathcal{D}$ be the conditional expectation with range $\mathcal{D}$ defined by

$$\mathcal{E}(x) = \sum_k X_{kk} \otimes |e_k\rangle\langle e_k| \tag{11}$$

and let $\mathcal{E}_*$ be the predual map on trace class operators with range

$$\mathcal{E}_*(\rho) = \sum_{k \in \mathcal{V}} \widetilde{\rho}_{kk} \otimes |e_k\rangle\langle e_k| \tag{12}$$

i.e., $\mathcal{E}_* : \mathcal{I}(\mathbf{h}) \to \mathcal{D}_*$ where

$$\mathcal{D}_* := \left\{ \sigma \in \mathcal{I}(\mathbf{h}); \sigma = \sum_{k \in \mathcal{V}} \widetilde{\sigma}_{kk} \otimes |e_k\rangle\langle e_k|, \widetilde{\sigma}_{kk} \in M_2(\mathbb{C}) \right\}.$$

By equality (10) and positivity of σ, it follows the positivity of $\widetilde{\sigma}_{pp} \in M_2(\mathbb{C})$ for all $p \in \mathcal{V}$. Moreover, $tr(\sigma) = 1$ if and only if $\sum_p tr(\widetilde{\sigma}_{pp}) = 1$, then

$$\sigma = \sum_{p \in \mathcal{V}} c_p \sigma_{pp} \otimes |e_p\rangle\langle e_p| + \sum_{p \neq q} \sigma_{pq} \otimes |e_q\rangle\langle e_p| \tag{13}$$

with $c_p := tr(\widetilde{\sigma}_{pp}) \geq 0$ for all $p \in \mathcal{V}$, $\sum_p c_p = 1$, $\sigma_{pp} := \frac{1}{c_p} \widetilde{\sigma}_{pp}$ (if $c_p > 0$) states in $\mathbb{C}^2$ and $\sigma_{pq} := \widetilde{\sigma}_{pq} \in M_2(\mathbb{C})$ for all $p \neq q$.

Therefore

$$\mathcal{D}_* = \left\{ \sigma \in \mathcal{I}(\mathbf{h}); \sigma = \sum_{p \in \mathcal{V}} c_p \sigma_{pp} \otimes |e_p\rangle\langle e_p|, \sum_p c_p = 1, \sigma_{pp} \in \mathcal{I}(\mathbb{C}^2) \right\}$$

and $\mathcal{I}(\mathbf{h}) = \mathcal{D}_* \oplus \mathcal{D}_{*off}$ with $\mathcal{D}_{*off} = \mathcal{E}_*^{\perp}(\mathcal{I}(\mathbf{h}))$, $\mathcal{E}_*^{\perp} := I - \mathcal{E}_*$.

Additionally, we obtain that if COQRWs has null-Hamiltonian, that is, his generator is given by (8) with $H = 0$ then

$$\mathcal{L}_*(\rho) = -\frac{1}{2} \sum_{j \neq m} (B_{jj}^* B_{jj} \rho_{jm} + \rho_{jm} B_{mm}^* B_{mm}) \otimes |e_j\rangle\langle e_m| \tag{14}$$

$$-\frac{1}{2}\sum_m c_m(B_{mm}^* B_{mm}\rho_{mm} + \rho_{mm} B_{mm}^* B_{mm}) \otimes |e_m\rangle\langle e_m| \tag{15}$$

$$+\sum_m \sum_{j\neq m} c_j B_{jm}\rho_{jj} B_{jm}^* \otimes |e_m\rangle\langle e_m| \tag{16}$$

hence $\mathcal{L}_*(\mathcal{D}_*) \subset \mathcal{D}_*$ y $\mathcal{L}_*(\mathcal{D}_{*off}) \subset \mathcal{D}_{*off}$ then $\mathcal{T}_{*t}(\mathcal{D}_*) \subset \mathcal{D}_*$ y $\mathcal{T}_{*t}(\mathcal{D}_{*off}) \subset \mathcal{D}_{*off}$ for all $t \geq 0$, i.e., $\mathcal{D}_*$ and $\mathcal{D}_{*off}$ are $\mathcal{T}_*$-invariants, equivalently

$$\mathcal{T}_{*t} \circ \mathcal{E}_* = \mathcal{E}_* \circ \mathcal{T}_{*t}, \quad \mathcal{T}_{*t} \circ \mathcal{E}_*^\perp = \mathcal{E}_*^\perp \circ \mathcal{T}_{*t} \tag{17}$$

for all $t \geq 0$.

4. Temporary Change in Quantum Correlations

In this section we measure temporary changes in quantum correlations using geometric quantum discord with Bures distance when COQRWs has null-Hamiltonian, that is, his generator is given by (8) with $H = 0$. In particular, we characterize the decay of correlations to zero, showing that turns out to be closely related with the structure of decoherence-free subspaces.

The Bures distances between states $\rho_1, \rho_2 \in \mathcal{I}(\mathsf{h})$ with $\mathsf{h} = \mathbb{C}^2 \otimes \mathbb{C}^d$ is given by $d_B(\rho_1,\rho_2) = \left[2\left(1 - \sqrt{F(\rho_1,\rho_2)}\right)\right]^{1/2}$, where $F(\rho_1,\rho_2)$ is the fidelity between ρ_1 and ρ_2, $F(\rho_1,\rho_2) = \left[tr\left(\left[\sqrt{\rho_2}\rho_1\sqrt{\rho_2}\right]^{1/2}\right)\right]^2$ and geometric quantum discord with Bures distance of a state $\rho \in \mathcal{I}(\mathsf{h})$ has been defined as

$$D_{\mathcal{D}_*}(\rho) := d_B(\rho,\mathcal{D}_*)^2 = \min_{\sigma\in\mathcal{D}_*} d_B(\rho,\sigma)^2 = 2\left(1 - \sqrt{F_{\mathcal{D}_*}(\rho)}\right)$$

with $F_{\mathcal{D}_*}(\rho) := \max_{\sigma\in\mathcal{D}_*} F(\rho,\sigma)$ (see [16],[17],[22],[23]).

The Bures distance can be used to bound from below and above the trace distance $d_1(\rho,\sigma) = tr(|\rho-\sigma|)$ (see [7]) as follows:

$$d_B(\rho,\sigma)^2 \leq d_1(\rho,\sigma) \leq \left[1 - \left(1 - \frac{1}{2}d_B(\rho,\sigma)^2\right)^2\right]^{1/2} \tag{18}$$

Theorem 6. *Let $\mathcal{T}$ be a COQRW with null-Hamiltonian then the following assertions are equivalent:*

(a)

$$\mathsf{h}_f = \bigcap_{m\in\mathcal{V}-\mathcal{V}_0} [(Ker(B_{mm})\otimes\mathbb{C}^d)\oplus(\mathbb{C}^2\otimes\{e_m\}^{\perp})]$$

with at most a unique $m\in\mathcal{V}-\mathcal{V}_0$ such that $Ker(B_{mm})\neq 0$.

(b) $\lim_{t\to 0} D_{\mathcal{D}_*}(\mathcal{T}_{*t}(\rho)) = 0$ *for all state ρ.*

(c) $\{\rho; supp(\rho)\subset \mathsf{h_f}\}\subset\{\rho;\mathcal{T}_{*t}(\rho)=\rho,$ *for all* $t\geq 0\}\subset\mathcal{D}_*$.

Proof. First note that $B^*_{mm}B_{mm}$ are positive-semidefinite matrices then there exist unitary matrices U_m such that

$$B^*_{mm}B_{mm} = U_m D_m U^*_m$$

with D_m diagonal matrix where his elements $(d^i_m)_{i=1,2}$ are non-negative.

Let $m\neq j$, with $m,j\in\mathcal{V}$,$\varrho\otimes|e_m\rangle\langle e_j|\in\mathcal{D}_{*off}$. Using equation (14) we obtain that

$$\begin{aligned}\mathcal{L}_*(\varrho\otimes|e_m\rangle\langle e_j|) &= -\frac{1}{2}(B^*_{mm}B_{mm}\varrho+\varrho B^*_{jj}B_{jj})\otimes|e_m\rangle\langle e_j| \\ &= -\frac{1}{2}(U_mD_mU^*_m\varrho+\varrho U_jD_jU^*_j)\otimes|e_m\rangle\langle e_j|.\end{aligned}$$

In particular, taking $\varrho = U_mZU^*_j$ for an arbitrary operator $Z=(Z_{rs})_{r,s=1,2}\in M_2(\mathbb{C})$, we obtain

$$\begin{aligned}\mathcal{L}_*(\varrho\otimes|e_m\rangle\langle e_j|) &= -\frac{1}{2}(U_mD_mZU^*_j+U_mZD_jU^*_j)\otimes|e_m\rangle\langle e_j| \\ &= U_m\left[-\frac{1}{2}(D_mZ+ZD_j)\right]U^*_j\otimes|e_m\rangle\langle e_j|.\end{aligned}$$

and recursively, we get

$$\mathcal{L}^n_*(\varrho\otimes|e_m\rangle\langle e_j|) = U_m\delta^{(n)}_{jm}(Z)U^*_j\otimes|e_m\rangle\langle e_j| \quad \text{for all } n\in\mathbb{N}, \tag{19}$$

where $\delta_{jm}(Z) := -\frac{1}{2}(D_mZ+ZD_j)$ and $\delta^{(n)}_{jm} = \underbrace{\delta_{jm}\circ\delta_{jm}\circ\ldots\circ\delta_{jm}}_{n\text{ times}}$. Hence

$$\mathcal{T}_{*t}(\varrho\otimes|e_m\rangle\langle e_j|) = U_m e^{t\delta_{jm}}(U^*_m\varrho U_j)U^*_j\otimes|e_m\rangle\langle e_j|.$$

Since $\delta_{jm}(Z) = -\frac{1}{2}(D_mZ+ZD_j) = (-1/2(d^p_m+d^q_j)Z_{pq})_{pq}$ then $\delta^{(n)}_{jm}(Z) = ((-1/2)^n(d^p_m+d^q_j)^nZ_{pq})_{pq}$ and therefore

$$e^{t\delta_{jm}}(Z) = (\exp[(-1/2)t(d^p_m+d^q_j)]Z_{pq})_{pq} \quad \text{for all } t\geq 0. \tag{20}$$

By (4) and (20), we obtain that

$$\mathcal{T}_{*t}(\varrho \otimes |e_m\rangle\langle e_j|) = U_m \left[\left(e^{-\frac{1}{2}t(d_m^p + d_j^q)} (U_m^* \varrho U_j)_{pq} \right)_{pq} \right] U_j^* \otimes |e_m\rangle\langle e_j| \qquad (21)$$

with $d_m^p \geq 0, d_j^q \geq 0$ for all $m, j \in \mathcal{V}$, $m \neq j$ and $p, q \in \{1, 2\}$, for any $\varrho \in M_2(\mathbb{C})$ and $t \geq 0$.

If we suppose that

$$\mathsf{h}_f = \bigcap_{m \in \mathcal{V} - \mathcal{V}_0} [(Ker(B_{mm}) \otimes \mathbb{C}^d) \oplus (\mathbb{C}^2 \otimes \{e_m\}^\perp)]$$

with at most a unique $m \in \mathcal{V} - \mathcal{V}_0$ such that $Ker(B_{mm}) \neq 0$, then $Ker(B_{mm}^* B_{mm}) \neq 0$ (by equation (7) and proposition 3) with $B_{mm}^* B_{mm} \neq 0$ ($m \in \mathcal{V} - \mathcal{V}_0$) therefore exists at most a unique $m \in \mathcal{V}$ such that $B_{mm}^* B_{mm}$ can have at most one zero eigenvalue, i.e.,exists at most a unique $m \in \mathcal{V}$ and exists at most a unique $p \in \{1, 2\}$ such that $d_m^p = 0$ and $d_j^q > 0$ for all $(j, q) \neq (m, p)$. By (21) it follows that

$$\lim_{t \to \infty} \mathcal{T}_{*t}(\varrho \otimes |e_m\rangle\langle e_j|) = 0 \quad \text{for any } \varrho \in M_2(\mathbb{C}),\ m, j \in \mathcal{V},\ m \neq j,$$

equivalently

$$\lim_{t \to \infty} \|\mathcal{T}_{*t}(\varrho \otimes |e_m\rangle\langle e_j|)\| = 0 \quad \text{for any } \varrho \in M_2(\mathbb{C}),\ m, j \in \mathcal{V},\ m \neq j, \qquad (22)$$

where $\| \cdot \|$ is the operator norm.

Give a arbitrary state

$$\rho = \sum_{k \in \mathcal{V}} c_k \rho_{kk} \otimes |e_k\rangle\langle e_k| + \sum_{k \neq r} \rho_{kr} \otimes |e_r\rangle\langle e_k|,$$

then

$$\mathcal{E}^\perp(\rho) = \sum_{k \neq r} \rho_{kr} \otimes |e_r\rangle\langle e_k|.$$

Using (17), (18) and (22), we see that

$$\begin{aligned} D_{\mathcal{D}_*}(\mathcal{T}_{*t}(\rho)) &\leq d_B(\mathcal{T}_{*t}(\rho), \mathcal{E}_*(\mathcal{T}_{*t}(\rho)))^2 \leq d_1(\mathcal{T}_{*t}(\rho), \mathcal{E}_*(\mathcal{T}_{*t}(\rho))) \\ &= d_1(\mathcal{T}_{*t}(\rho), \mathcal{T}_{*t}(\mathcal{E}_*(\rho))) = tr(\mathcal{T}_{*t}(\mathcal{E}_*^\perp(\rho))) \\ &= \sum_{k \neq r} tr(\mathcal{T}_{*t}(\rho_{kr} \otimes |e_r\rangle\langle e_k|)) \end{aligned}$$

$$\leq 2d \sum_{k\neq r} \|\mathcal{T}_{*t}(\rho_{kr} \otimes |e_r\rangle\langle e_k|)\| \xrightarrow[t\to\infty]{} 0$$

then $(a) \Rightarrow (b)$.

Now, If we suppose that $\lim_{t\to 0} D_{\mathcal{D}_*}(\mathcal{T}_{*t}(\rho)) = 0$ for all state ρ. In particular, if we take ρ such that $\mathcal{T}_{*t}(\rho) = \rho$ for all $t \geq 0$ then $D_{\mathcal{D}_*}(\rho) = 0$, therefore $\rho \in \overline{\mathcal{D}_*}^{d_B}$ where $\overline{\mathcal{D}_*}^{d_B}$ is the closure of $\mathcal{D}_*$ with respect to Bures distance. By inequality (18), it is easy to see $\overline{\mathcal{D}_*}^{d_B} = \mathcal{D}_*$ and that

$$\{\rho; supp(\rho) \subset \mathsf{h_f}\} \subset \{\rho; \mathcal{T}_{*t}(\rho) = \rho, \text{ for all } t \geq 0\}$$

therefore $\rho \in \mathcal{D}_*$, then $(b) \Rightarrow (c)$.

If $\{\rho; supp(\rho) \subset \mathsf{h_f}\} \subset \{\rho; \mathcal{T}_{*t}(\rho) = \rho, \forall t \geq 0\} \subset \mathcal{D}_*$ and exists $m, j \in \mathcal{V} - \mathcal{V}_0$, $m \neq j$ such that $Ker(B_{mm}) \neq 0$ and $Ker(B_{jj}) \neq 0$ then exist d_m^r eigenvalue of $B_{mm}^* B_{mm}$ and d_j^s eigenvalue of $B_{jj}^* B_{jj}$ such that $d_m^r = d_j^s = 0$, $d_m^l > 0$ if $l \neq r$, and $d_j^k > 0$ if $k \neq s$. Using (21), we see that

$$\mathcal{T}_{*t}(\varrho \otimes |e_m\rangle\langle e_j|) = X + Y_t$$

$$X = U_m[(U_m^* \varrho U_j)_{rs} |f_r\rangle\langle f_s| + (U_m^* \varrho U_j)_{sr} |f_s\rangle\langle f_r|] U_j^* \otimes |e_m\rangle\langle e_j|, \tag{23}$$

$$\begin{aligned} Y_t = U_m[&\left(e^{-\frac{1}{2}t(d_m^l + d_j^k)} (U_m^* \varrho U_j)_{lk}\right)_{lk} |f_l\rangle\langle f_k| \\ &+ \left(e^{-\frac{1}{2}t(d_m^l + d_j^k)} (U_m^* \varrho U_j)_{kl}\right)_{kl} |f_k\rangle\langle f_l|] U_j^* \otimes |e_m\rangle\langle e_j|, \end{aligned}$$

where $\mathcal{T}_{*t}(X) = X$ for all $t \geq 0$ and $\lim_{t\to\infty} Y_t = 0$. We take ϱ such that

$$(U_m^* \varrho U_j)_{rs} \neq 0 \text{ and } (U_m^* \varrho U_j)_{sr} \neq 0 \tag{24}$$

then

$$\begin{aligned} X = \lim_{t\to\infty} \mathcal{T}_{*t}(\varrho \otimes |e_m\rangle\langle e_j|) &\in \mathcal{D}_{oof*} \cap \{\rho; \mathcal{T}_{*t}(\rho) = \rho, \forall t \geq 0\} \\ &\subset \mathcal{D}_{oof*} \cap \mathcal{D}_* = 0. \end{aligned}$$

By (23), we obtain that $(U_m^* \varrho U_j)_{sr} = (U_m^* \varrho U_j)_{rs} = 0$. This is a contradiction with (24) . Therefore exist at most a unique $m \in \mathcal{V} - \mathcal{V}_0$ such that $Ker(B_{mm}) \neq 0$, then $(c) \Rightarrow (a)$. □

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