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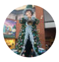

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Gaussian Quantum Markov Semigroups on a One-Mode Fock Space: Irreducibility and Normal Invariant States

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Abstract. We consider the most general Gaussian quantum Markov semigroup on a one-mode Fock space, discuss its construction from the generalized GKSL representation of the generator. We prove the known explicit formula on Weyl operators, characterize irreducibility and its equivalence to a Hörmander type condition on commutators and establish necessary and sufficient conditions for existence and uniqueness of normal invariant states. We illustrate these results by applications to the open quantum oscillator and the quantum Fokker-Planck model.

Keywords: Quantum Markov semigroup, quasi-free semigroup, GKSL generator, Gaussian, irreducibility.

1. Introduction

Gaussian semigroups are of utmost importance in many fields because they arise in several relevant models, they form a class with a rich structure allowing one to establish and take advantage of a number of explicit formulas. This happens also in quantum theory of open systems with quantum Markov semigroups (QMS), namely weakly*-continuous semigroups $(\mathcal{T}_t)_{t\geq 0}$ of completely positive, identity preserving, normal maps on a von Neumann algebra. When this is the algebra $\mathcal{B}(\Gamma(\mathbb{C}^d))$ of all bounded operators on the Fock space $\Gamma(\mathbb{C}^d)$ a QMS is called *Gaussian* if the predual semigroup $(\mathcal{T}_{*t})_{t\geq 0}$ acting on trace class operators on $\Gamma(\mathbb{C}^d)$ preserves Gaussian states (see Sect. 5 for the definition).

There are two typical approaches to this class of semigroups. Physicists usually consider generators represented in a Gorini-Kossakowski-Sudharshan-Lindblad [19, 22] (GKSL) form (see (1), (3), (2) below), which is only formal because it involves unbounded operators, and compute moments of Gaussian observables without concern about the existence and well-definiteness of the dynamics (see, for instance [1, 3, 4, 20, 23] and the references therein). Mathematicians introduce Gaussian QMSs by their action on Weyl operators (see, e.g., [15, 30]) of a regular representation of canonical commutation relations (CCR) but they just show ([30] Proposition 4.8, Theorem 4.9) that the action of generator, on a certain restricted domain, admits a generalized GKSL representation with unbounded Hamiltonian and noise operators.

The joining link for handling both techniques and exploiting the advantages of each one of them is the characterization of the unbounded generator with a generalized GKSL form involving unbounded operators that are either linear or quadratic in creation and annihilation operators. In this way one can go beyond explicit computations on Gaussian states and observables and study, for instance, the evolution of any initial state applying general results from the theory of QMS.

In this paper we consider the most general Gaussian quantum Markov semigroup on the one mode Fock space $\Gamma(\mathbb{C})$ of the regular representation of one-dimensional CCR. First we discuss its construction starting from the unbounded generator in its generalized GKSL form and give a proof of the known explicit formula for the action on Weyl operators. Second, we fully characterize irreducibility in terms of parameters of the model. This is an important property of the dynamics because it implies that the system has to be regarded as a whole and reduction to subsystems is not possible. In particular, the support of any initial state cannot remain confined in a proper subspace (see, e.g., [18]). Third, still in terms of these parameters, establish necessary and sufficient conditions for existence and uniqueness of normal invariant states. As a corollary, for any initial state, we also deduce convergence towards the unique invariant state.

In this way, we provide a unified treatment of both approaches and a thorough study of the one-dimensional case. This is quite complex because it depends on many parameters and a detailed (somewhat lengthy) analysis of several special subcases is necessary.

Gaussian QMSs on the von Neumann algebra of all bounded operators on the one-mode Fock space $\Gamma(\mathbb{C})$ are uniquely defined by pre-generators in a generalized (GKSL) form

$$\mathcal{L}(x) \;=\; \mathrm{i}\,[H,x] - \frac{1}{2}\sum_{\ell=1}^{2}\left(L_\ell^* L_\ell\, x - 2L_\ell^* x L_\ell + x\, L_\ell^* L_\ell\right), \tag{1}$$

where unbounded operators L_1, L_2, called noise operators or Kraus operators, depend linearly on Bosonic creation and annihilation operators on $\Gamma(\mathbb{C})$

$$L_\ell \;=\; \overline{v}_\ell a + u_\ell a^\dagger \qquad \overline{v}_\ell, u_\ell \in \mathbb{C}\,, \quad \ell = 1,2\,, \tag{2}$$

and H is the operator

$$H \;=\; \Omega\, a^\dagger a + \frac{\kappa}{2} a^{\dagger 2} + \frac{\overline{\kappa}}{2} a^2 + \frac{\zeta}{2} a^\dagger + \frac{\overline{\zeta}}{2} a \tag{3}$$

with $\Omega \in \mathbb{R}, \kappa, \zeta \in \mathbb{C}$. The operators L_1, L_2 will be assumed to be linearly independent. However, we shall also consider the special case when L_2 does not appear in (1), namely the multiplicity of the completely positive part of (1) is one. We will not consider when there are no noise operators L_ℓ, corresponding to a closed system.

In this paper, we find three main new results. The first one is the complete characterization of irreducibility and its equivalence to a Hörmander type condition on certain commutators. More precisely, we show that the QMS with pre-generator (1) is irreducible if the completely positive part has multiplicity two, namely the operators L_1, L_2 are linearly independent (Theorems 5). While, in the case where the completely positive part has multiplicity one (i.e., formally, L_2 does not appear in (1)), we show (Theorem 7) that the QMS is irreducible if and only if the operators L_1 and $[H, L_1]$ are linearly independent. This is clearly a Hörmader type multiple commutator condition in which one needs only the first order commutator because of one-dimensionality of the CCR. Here, however, Hörmander condition is established for a differential operator with second order anti-selfadjoint part, not first order as in the classical case.

The second main result is the characterization of Gaussian QMSs with normal invariant states by two simple inequalities on the parameters of the model. We show (Theorems 8, 9) that one can find a normal invariant state, which is explicit and is a quantum Gaussian state, if and only if

$$\gamma \;=\; \frac{1}{2}\sum_{\ell=1,2} \left(|v_\ell|^2 - |u_\ell|^2\right) \;>\; 0 \quad \text{and} \quad \gamma^2 + \Omega^2 - |\kappa|^2 \;>\; 0\,.$$

Note that normal invariant states may exist also when the Hamiltonian has no eigenvalues, however transitions to lower-level states induced by the dissipative part must be stronger to compensate the effect of the Hamiltonian H without eigenstates (see Remark after Theorem 8). The third main result is uniqueness of Gaussian invariant states in the set of *all* normal invariant states and convergence towards invariant states (Theorem 8) which follows from irreducibility in most cases.

The paper is organized as follows. In Sect. 2 we begin by describing the generator in the generalized GKSL form and construct Gaussian QMSs by the minimal semigroup method. After proving Markovianity (i.e., preservation of the identity operator) in Theorem 1 we have a characterization of its domain at our disposal that we exploit for proving the known explicit formula for the action on Weyl operators (Theorem 2).

Then we turn our attention to irreducibility. In Sect. 3 we prove that it always holds when the completely positive part of the generator has two noise operators (Theorem 5). The case in which there is only one noise operator where the Hörmander type commutator condition appears is studied in Sect. 4 (see the decision tree at the end of the section). In Sect. 6 we study normal invariant states showing that, when they exist, they are also unique. Moreover, it turns out that they are either faithful or pure (Proposition 5). Finally, we illustrate these results by applications to the open quantum oscillator and the quantum Fokker-Planck model.

2. Gaussian QMSs

In this section we introduce the class of QMS that we will analyze in this paper.

Let $\mathsf{h} = \Gamma(\mathbb{C})$ be the Fock space on $\mathbb{C}$ with canonical orthonormal basis $(e_n)_{n\geq 0}$. Each vector e_n is called n-particle vector. For each $z \in \mathbb{C}$ the vector

$$e(z) \;=\; \sum_{n\geq 0} \frac{z^n}{\sqrt{n!}}\, e_n$$

is called exponential or coherent vector with parameter z. The vector space of finite linear combinations of vectors of the canonical orthonormal basis, denoted by D, is a natural common domain for all unbounded operators that we will consider. One could consider as domain D the linear span of exponential vectors, or the linear span of $(e_n)_{n\geq 0}$ together with exponential vectors, without further complications.

The number operator is the selfadjoint operator on h defined by

$$\mathrm{Dom}(N) \;=\; \Big\{ \xi = \sum_{n\geq 0} \xi_n e_n \in \mathsf{h} : \sum_{n\geq 0} n^2 |\xi_n|^2 < \infty \Big\} \qquad N\xi \;=\; \sum_{n\geq 0} n\, \xi_n e_n\,.$$

Annihilation and creation operators on h are defined on the domain $\mathrm{Dom}(N^{1/2})$ by

$$a\,\xi \;=\; \sum_{n\geq 1} \sqrt{n}\, \xi_n\, e_{n-1}\,, \qquad a^\dagger \xi \;=\; \sum_{n\geq 1} \sqrt{n+1}\, \xi_n\, e_{n+1}\,.$$

It is not difficult to see that $a, a^\dagger$ are closed operators and they are mutually adjoint. Alternatively, they can be defined via polar decomposition $a^\dagger = S(N+\mathbb{1})^{1/2}$, $a = S^* N^{1/2}$ where $\mathbb{1}$ is the identity operator and S the right shift defined by $Se_n = e_{n+1}$.

Annihilation and creation operators satisfy the canonical commutation relation (CCR)

$$[a, a^\dagger] = \mathbb{1}$$

on $\mathrm{Dom}(N)$, $[\,\cdot\,,\,\cdot\,]$ denoting the commutator. Moreover, for all $z \in \mathbb{C}$ the operator with domain $\mathrm{Dom}(N^{1/2})$ defined by $za^\dagger - \overline{z}a$ is anti-selfadjoint an one can define the unitary Weyl operator

$$W(z) = \exp(za^\dagger - \overline{z}a)\,.$$

It is not difficult to see as well that exponential vectors belong to the domain of N^k for all $k \geq 0$. In particular, they belong to the domain of a, $a^\dagger$ and

$$a\, e(z) = z e(z)\,, \qquad a^\dagger e(z) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} e(z+\varepsilon)\Big|_{\varepsilon=0}\,.$$

In this paper we are concerned with quantum Markov semigroups (QMS) with pre-generators in a generalized Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) form (1). The domain of $\mathcal{L}$ will be described below in Theorem 1, after the construction of a QMS by the minimal semigroup method. The operators L_1, L_2 are defined on $\mathrm{Dom}(N^{1/2})$ and H is the operator on $\mathrm{Dom}(N)$ defined by (3). The operator L_1, L_2, also called noise or Kraus operators, will be assumed linearly independent, namely

$$\overline{v}_1 u_2 - \overline{v}_2 u_1 \neq 0\,,$$

so as to consider a generalized GKSL representation of $\mathcal{L}$ with the minimum number of Kraus operators. We shall consider also the special case when there is only one Kraus operator L_1 but not the “reversible”, purely Hamiltonian, case with no Kraus operator.

As shown in [13, Proposition 4.9] the operator G defined by closure of the operator $-\mathrm{i}H - (L_1^* L_1 + L_2^* L_2)/2$ defined on D by

$$\begin{aligned} G \;=\; & -\Big(\frac{1}{2}\left(|v_1|^2 + |v_2|^2 + |u_1|^2 + |u_2|^2\right) + \mathrm{i}\Omega\Big) a^\dagger a - \frac{1}{2}\left(|u_1|^2 + |u_2|^2\right) \mathbb{1} \\ & -\frac{1}{2}\left(v_1 u_1 + v_2 u_2 - \mathrm{i}\kappa\right) a^{\dagger 2} - \frac{1}{2}\left(\overline{v}_1 \overline{u}_1 + \overline{v}_2 \overline{u}_2 + \mathrm{i}\overline{\kappa}\right) a^2 - \frac{\mathrm{i}}{2}\Big(\zeta a^\dagger + \overline{\zeta} a\Big) \end{aligned}$$

generates a strongly continuous semigroup $(P_t)_{t\geq 0}$ on h, therefore we can construct the minimal QMS associated with operators G, L_1, L_2.

We briefly recall the construction (see [13, Sect. 3.3]). Let $x \in \mathcal{B}(\mathsf{h})$ and $t \geq 0$ and define non decreasing sequence of completely positive maps $\mathcal{T}_t^{(n)}$ on $\mathcal{B}(\mathsf{h})$ by $\mathcal{T}_t^{(0)}(x) = P_t^* x P_t$ and

$$\begin{aligned} \Big\langle \xi', \mathcal{T}_t^{(n+1)}(x)\xi \Big\rangle \;=\; & \left\langle P(t)\xi', xP(t)\xi \right\rangle \\ & + \sum_{\ell=1}^{2} \int_0^t \Big\langle L_\ell P(t-s)\xi', \mathcal{T}_s^{(n)}(x) L_\ell P(t-s)\xi \Big\rangle \,\mathrm{d}s \end{aligned}$$

where $\xi, \xi' \in D$. It can be shown that one can define a weak* continuous semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of normal completely positive maps on $\mathcal{B}(\mathsf{h})$ by $\mathcal{T}_t(x) = \sup_{n\geq 0} \mathcal{T}_t^{(n)}(x)$ for all x positive and then extend to an arbitrary $x \in \mathcal{B}(\mathsf{h})$ by decomposition as sum of positive operators.

THEOREM 1 *The minimal QMS $\mathcal{T}$ is identity preserving. The domain of its generator $\mathcal{L}$ consists of the set of $x \in \mathcal{B}(\mathsf{h})$ for which the quadratic form with domain $D \times D$*

$$\pounds(x)[\xi', \xi] \;=\; \left\langle G\xi', x\xi \right\rangle + \sum_{\ell=1}^{2} \left\langle L_\ell \xi', x L_\ell \xi \right\rangle + \left\langle \xi', x\, G\xi \right\rangle$$

is bounded. Moreover, $\mathcal{T}$ is the unique weak-continuous semigroup of positive operators on $\mathcal{B}(\mathsf{h})$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \xi', \mathcal{T}_t(x)\xi \right\rangle \Big|_{t=0} \;=\; \pounds(x)[\xi', \xi]$$

for all $x \in \mathcal{B}(\mathsf{h})$, $\xi', \xi \in \mathrm{Dom}(G)$.

Proof. $\mathcal{T}$ is identity preserving by Proposition 4.12 in [13]. The characterization of the domain of the generator $\mathcal{L}$ is given in [13] Proposition 3.33. Finally, if $(\mathcal{T}_t')_{t\geq 0}$ is another such semigroup then, for x positive, we can prove inductively that $\mathcal{T}_t'(x) \geq \mathcal{T}_t^{(n)}(x)$ for all $n \geq 0$ and $t \geq 0$, therefore $\mathcal{T}_t'(x) \geq \mathcal{T}_t(x)$. Considering the operator $x = \|x\|\mathbb{1} - x$, which is positive, we have also

$$\|x\|\mathbb{1} - \mathcal{T}_t'(x) \;=\; \mathcal{T}_t'(\|x\|\mathbb{1} - x) \;\geq\; \mathcal{T}_t(\|x\|\mathbb{1} - x) \;=\; \|x\|\mathbb{1} - \mathcal{T}_t(x)$$

therefore $\mathcal{T}_t(x) = \mathcal{T}_t'(x)$. □

2.1. Explicit formula on Weyl operators

An interesting known feature of Gaussian QMSs is the explicit formula for their action on the dense subalgebra of Weyl operators (see [24, 25, 30] and also [27, 31]). We will now present this formula and establish the relationship with the GKSL generator (1).

We begin by recalling some useful formulae on the action of Weyl operators

$$W(z)e(f) \;=\; \exp\left(-|z|^2/2 - \overline{z}f\right) e(f+z)$$

A straightforward computation on exponential vectors yields the Weyl commutation relations

$$W(z)W(z') \;=\; \exp\left(-\mathrm{i}\Im(\overline{z}z')\right) W(z+z')$$

in particular $W(z)^* = W(-z)$. Moreover, we have

$$W(z)^* a\, W(z) \;=\; a + z\mathbb{1}\,, \qquad W(z)^* a^\dagger W(z) \;=\; a^\dagger + \overline{z}\mathbb{1}\,.$$
$$[a, W(z)] \;=\; z\, W(z)\,, \qquad [a^\dagger, W(z)] \;=\; \overline{z}\, W(z)\,.$$

The following Theorem gives the explicit action of maps $\mathcal{T}_t$ on Weyl operators. In the sequel $\Re(z)$ and $\Im(z)$ denote the real and imaginary part of a complex number z.

THEOREM 2 *Let $(\mathcal{T}_t)_{t\geq 0}$ be the QMS with generalized GKSL generator associated with H, L_1, L_2 as in (2). For all Weyl operator $W(z)$ we have*

$$\mathcal{T}_t(W(z)) = \exp\Big(-\frac{1}{2}\int_0^t \Re\left(\overline{e^{sZ}z}\, C e^{sZ} z\right) \mathrm{d}s + \mathrm{i}\int_0^t \Re\left(\overline{\zeta}\, e^{sZ} z\right) \mathrm{d}s\Big) W\left(e^{tZ} z\right), \tag{4}$$

where Z and C are the real linear operators

$$Zz \;=\; \Big(\mathrm{i}\Omega + \sum_{\ell=1}^{2}(|u_\ell|^2 - |v_\ell|^2)/2\Big) z + \mathrm{i}\kappa\overline{z}\,, \tag{5}$$

$$Cz \;=\; \sum_{\ell=1}^{2}\left((|u_\ell|^2 + |v_\ell|^2) z + 2 v_\ell u_\ell\, \overline{z}\right). \tag{6}$$

Proof. One can check [30, Theorem 3.1] the semigroup law $\mathcal{T}_t(\mathcal{T}_s(W(z))) = \mathcal{T}_{t+s}(W(z))$ for all $t, s \geq 0$. The derivative of $\langle e(g), \mathcal{T}_t(W(z)) e(f)\rangle$ at time $t = 0$, namely $\mathcal{L}(W(z))[e(g), e(f)]$ is equal to

$$\begin{aligned}
&\mathrm{i}\left(\langle He(g), W(z)e(f)\rangle - \langle e(g), W(z)He(f)\rangle\right) \\
&\quad - \;\frac{1}{2}\sum_{\ell=1}^{2}\Big(\langle L_\ell^* L_\ell e(g), W(z)e(f)\rangle - 2\,\langle L_\ell e(g), W(z) L_\ell e(f)\rangle \\
&\qquad\qquad + \;\langle e(g), W(z) L_\ell^* L_\ell e(f)\rangle \Big) \\
&\quad = \;\mathrm{i}\,\langle e(g), [H, W(z)] e(f)\rangle + \frac{1}{2}\sum_{\ell=1}^{2}\Big(\langle e(g), L_\ell^*[W(z), L_\ell] e(f)\rangle \\
&\qquad + \;\langle e(g), [L_\ell^*, W(z)] L_\ell e(f)\rangle \Big)\,,
\end{aligned}$$

where all operator compositions make sense because exponential vectors are in the domain of any power of the number operator. Computing the commutators

$$[W(z), L_\ell] \;=\; -\left(\overline{v_\ell} z + u_\ell \overline{z}\right) W(z)\,, \quad [L_\ell^*, W(z)] \;=\; \left(\overline{u_\ell} z + v_\ell \overline{z}\right) W(z)$$

$$
\begin{aligned}
[H, W(z)] &= W(z)\Big((\Omega\overline{z}+\overline{\kappa}z)\,a + (\Omega z+\kappa\overline{z})\,a^{\dagger}\Big) \\
&+ \Big(\Omega|z|^2 + \frac{\kappa\overline{z}^2}{2} + \frac{\overline{\kappa}z^2}{2} + \frac{\zeta\,\overline{z}}{2} + \frac{\overline{\zeta}z}{2}\Big)W(z) \\
[L_\ell^*, [W(z), L_\ell]] &= -\left(\left(|u_\ell|^2+|v_\ell|^2\right)|z|^2 + \overline{v_\ell u_\ell}z^2 + v_\ell u_\ell \overline{z}^2\right)W(z)
\end{aligned}
$$

we can write $\mathcal{L}(W(z))[e(g), e(f)]$ as $\langle e(g), W(z)X(z)e(f)\rangle$ where

$$
\begin{aligned}
X(z) &= \Big(\left(\left(|u_\ell|^2-|v_\ell|^2\right)z/2 + \mathrm{i}\,(\Omega z+\kappa\overline{z})\right)a^{\dagger} \\
&\quad - \left(\left(|u_\ell|^2-|v_\ell|^2\right)\overline{z}/2 - \mathrm{i}\,(\Omega\overline{z}+\overline{\kappa}z)\right)a\Big) \\
&- \frac{1}{2}\left((|u_\ell|^2+|v_\ell|^2)|z|^2 + \overline{v_\ell u_\ell}z^2 + v_\ell u_\ell\overline{z}^2\right) \\
&\quad + \mathrm{i}\Big(\Omega|z|^2 + \frac{\kappa\overline{z}^2+\overline{\kappa}z^2}{2}\Big) + \mathrm{i}\frac{\zeta\,\overline{z}+\overline{\zeta}z}{2}\,.
\end{aligned}
$$

Since exponential vectors belong to the domain of $a, a^{\dagger}$, compute now the derivative of $W(e^{tZ}z)e(f)$ at time $t=0$ as follows

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}W(e^{tZ}z)e(f)\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\exp\Big(-\frac{1}{2}\left|e^{tZ}z\right|^2 - \overline{e^{tZ}z}f\Big)e(e^{tZ}z+f)\Big|_{t=0} \\
&= \left(-\Re\left(\overline{z}\,Zz\right) - \overline{Zz}f\right)\exp\left(-\frac{1}{2}\left|z\right|^2 - \overline{z}f\right)e(z+f) \\
&+ \exp\Big(-\frac{1}{2}\left|z\right|^2 - \overline{z}f\Big)\frac{\mathrm{d}}{\mathrm{d}t}e(e^{tZ}z+f)\Big|_{t=0} \\
&= -W(z)\left(\overline{Zz}\,a + \Re\left(\overline{z}\,Zz\right)\right)\,e(f) \\
&+ \exp\Big(-\frac{1}{2}\left|z\right|^2 - \overline{z}f\Big)\frac{\mathrm{d}}{\mathrm{d}t}(e^{tZ}z+f)\Big|_{t=0}a^{\dagger}e(z+f)\,.
\end{aligned}
$$

Recalling the commutation relation $[a^{\dagger}, W(z)] = \overline{z}W(z)$ we find

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}W(e^{tZ}z)e(f)\Big|_{t=0} &= -W(z)\left(\overline{Zz}\,a + \Re\left(\overline{z}\,Zz\right)\right)\,e(f) + (Zz)a^{\dagger}W(z)e(f) \\
&= W(z)(Zz\,a^{\dagger} - \overline{Zz}\,a - \Re\left(\overline{z}\,Zz\right) + \overline{z}\,Zz)e(f) \\
&= W(z)(Zz\,a^{\dagger} - \overline{Zz}\,a + \frac{1}{2}\left(\overline{z}\,Zz - \overline{Zz}\,z\right))e(f)\,.
\end{aligned}
$$

Computing the derivative of the exponential factor in (4) at $t=0$, the derivative of the scalar product of the right-hand side of (4) with two exponential vectors $e(g), e(f)$ can be written as well as $\langle e(g), W(z)Y(z)e(f)\rangle$ where $Y(z)$ is

$$
Y(z) = (Zz\,a^{\dagger} - \overline{Zz}\,a) + \frac{1}{2}\left(\overline{z}\,Zz - \overline{Zz}\,z\right) - \frac{1}{2}\Re\left(\overline{z}\,Cz\right) + \mathrm{i}\Re\left(\overline{\zeta}z\right).
$$

The conclusion follows form $X(z) = Y(z)$ for all $z\in\mathbb{C}$. □

Remark 1 If the GKSL generator has only one Kraus operator L_1 formula (4) also holds with real linear operators Z and C, formally defined in the same way, setting $v_2 = u_2 = 0$ in (5) and (6).

3. Irreducibility: The Case of Two Noise Operators L_1, L_2

In the study of the evolution of an open quantum system irreducibility plays a key role because it guarantees that there is no proper subsystem which is invariant under the evolution. Therefore the system has to be regarded as a whole and reduction to subsystems is not possible. In addition, irreducibility is a key assumption of many results on the asymptotic behaviour of QMS (see [16]) and irreducible subsystems constitute the building blocks in the analysis of the structure of normal invariant states of a QMS (see [10]).

In this section we show that the Gaussian QMS with two linearly independent noise operators L_1, L_2 is irreducible. Gaussian QMS with only one operator L will be considered in Sect. 4.

DEFINITION 1 A QMS $\mathcal{T}$ on $\mathcal{B}(\mathsf{h})$ is called *irreducible* if there exists no non-trivial orthogonal projection p on h such that $\mathcal{T}_t(p) \geq p$ for all $t \geq 0$.

A projection p such that $\mathcal{T}_t(p) \geq p$ for all $t \geq 0$ is called *subharmonic* following the terminology in use in the classical theory of Markov processes. The following result (see Theorem III.1 in [14]) characterizes such projections.

THEOREM 3 *A projection p is subharmonic for $\mathcal{T}$ if and only if the range* $\mathrm{Rg}(p)$ *of p is invariant for the operators P_t $(t \geq 0)$ of the strongly continuous contraction semigroup on h generated by G and $L_\ell u = pL_\ell u$, for all $u \in$* $\mathrm{Dom}(G) \cap \mathrm{Rg}(p)$*, and all $\ell \geq 1$.*

It is worth noticing here that, by general results on strongly continuous semigroups (see [14] Lemma III.1), if $\mathrm{Rg}(p)$ is invariant for the operators P_t, then $\mathrm{Dom}(G) \cap \mathrm{Rg}(p)$ is dense in $\mathrm{Rg}(p)$ and so conditions on the operators L_ℓ are not reduced to the sole zero vector.

In view of this characterization of subharmonic projections, it is now intuitively clear that, if there are two linearly independent Kraus operators, the range of a subharmonic projection should be an invariant subspace for a and $a^\dagger$ and so it will be trivial by irreducibility of the Fock representation of the CCR. However, the necessary clarifications on operator domains are now in order.

Let G_0 be the closure of the operator $-(L_1^* L_1 + L_2^* L_2)/2$ defined on D which is symmetric. It is easy to check that every vector in D is an analytic vector for G_0. Therefore an application of Nelson's theorem on analytic vectors shows that it is selfadjoint. The following is the key result on the domain of the operator G that we need for proving irreducibility.

THEOREM 4 *If there are two linearly independent noise operator L_1, L_2 the domain of the operators G and G_0 coincide with the domain of the number operator N.*

We defer the proof to Appendix A and proceed to the main result of this section. Note that the property $\mathrm{Dom}(G) = \mathrm{Dom}(G_0) = \mathrm{Dom}(N)$ plays a key role in the proof.

THEOREM 5 *The QMS with generalized GKSL generator associated with H as in (3) and two linearly independent noise operator L_1, L_2 as in (2) is irreducible.*

Proof. Let $\mathcal{V}$ be a nonzero closed subspace of h which is invariant for the contraction operators P_t of the semigroup generated by G and $L_\ell\,(\mathrm{Dom}(G) \cap \mathcal{V}) \subseteq \mathcal{V}$ for $\ell = 1, 2$.

By the linear independence of L_1, L_2, since $\mathrm{Dom}(G) = \mathrm{Dom}(N)$ we have also

$$a\,(\mathrm{Dom}(N) \cap \mathcal{V}) \subseteq \mathrm{Dom}(N^{1/2}) \cap \mathcal{V} \qquad a^\dagger\,(\mathrm{Dom}(N) \cap \mathcal{V}) \subseteq \mathrm{Dom}(N^{1/2}) \cap \mathcal{V}$$
$$a^\dagger a\,(\mathrm{Dom}(N) \cap \mathcal{V}) \subseteq \mathcal{V} \qquad aa^\dagger\,(\mathrm{Dom}(N) \cap \mathcal{V}) \subseteq \mathcal{V}$$

hence, denoting by p the orthogonal projection onto $\mathcal{V}$,

$$p^\perp a p \;=\; 0 \;=\; p a p^\perp\,, \qquad p^\perp a^\dagger p \;=\; 0 \;=\; p a^\dagger p^\perp$$

on $\mathrm{Dom}(N) \cap \mathcal{V}$ and, left multiplying by $a^\dagger$ the first identity,

$$p^\perp a^\dagger a p \;=\; 0 \;=\; p a^\dagger a p^\perp.$$

It follows that, for all $\lambda > 0$, we have the commutation $(\lambda \mathbb{1} + N)\,p = p\,(\lambda \mathbb{1} + N)$ and, left and right multiplication by the resolvent $(\lambda \mathbb{1} + N)^{-1}$ yields

$$p\,(\lambda \mathbb{1} + N)^{-1} \;=\; (\lambda \mathbb{1} + N)^{-1}\,p\,.$$

In particular, for all $k > 0$, considering bounded Yosida approximations $N_k = kN\,(k\mathbb{1} + N)^{-1}$ of N that converge strongly to N on $\mathrm{Dom}(N)$ we have

$$p\,kN\,(k\mathbb{1} + N)^{-1} \;=\; kN\,(k\mathbb{1} + N)^{-1}\,p$$

and so $p\,e^{-tN_k} = e^{-tN_k}p$ for all $t, k > 0$. Taking the limit as $k \to +\infty$, by the Trotter-Kato theorem [12, Th. 4.8 p. 209] we find

$$p\,e^{-tN} \;=\; e^{-tN}p \quad \forall\, t \geq 0\,. \tag{7}$$

Let $v \in \mathcal{V}$, $v \neq 0$ with expansion in the canonical basis

$$v \;=\; \sum_{k \geq k_0} v_k e_k\,,$$

where k_0 is the minimum k for which $v_k \neq 0$. Clearly, by (7), $e^{-tN}v \in \mathcal{V}$ for all $t \geq 0$ and so

$$e^{k_0 t}e^{-tN}v \;=\; \sum_{k\geq k_0} e^{-(k-k_0)t}v_k e_k \;=\; v_{k_0}e_{k_0} + \sum_{k>k_0} e^{-(k-k_0)t}v_k e_k \in \mathcal{V}$$

for all $t \geq 0$. Taking the limit at $t \to +\infty$, we find $e_{k_0} \in \mathcal{V}$. Acting on e_{k_0} with operators a and $a^\dagger$ we can immediately show that every vector e_k of the basis belongs to $\mathcal{V}$ and the proof is complete. □

4. Irreducibility: The Case of a Single Noise Operator L

In this section we study the case where there is a single operator

$$L \;=\; \overline{v}a + ua^\dagger \quad \text{with} \quad v \neq 0 \quad \text{or} \quad u \neq 0\,.$$

This case is much more entangled. We begin by considering the algebraic aspect of the problem disregarding, for the moment, domain issues that will be considered later.

We are looking for common invariant subspaces for the operators G and L and so also for the commutator $[L, G]$. A straightforward computation yields

$$\begin{aligned} -2\,[L,G] \;&=\; [L, L^*L + 2\mathrm{i}H] \\ &=\; [L, L^*]\,L + 2\mathrm{i}\,(\overline{v}\Omega - u\overline{\kappa})\,a - 2\mathrm{i}\,(u\Omega - \overline{v}\kappa)\,a^\dagger + 2\mathrm{i}\left(\overline{v}\zeta - u\overline{\zeta}\right). \end{aligned} \tag{8}$$

Thus the candidate subspace must be invariant for the operators

$$G \;=\; -\frac{1}{2}L^*L - \mathrm{i}H\,, \quad L \;=\; \overline{v}a + ua^\dagger\,, \quad \widetilde{L} \;=\; (\overline{v}\Omega - u\overline{\kappa})\,a + (\overline{v}\kappa - u\Omega)\,a^\dagger\,.$$

If the operators L and $\widetilde{L}$ are linearly independent, namely

$$\det\begin{bmatrix} \overline{v}\Omega - u\overline{\kappa} & \overline{v}\kappa - u\Omega \\ \overline{v} & u \end{bmatrix} \;\neq\; 0\,, \tag{9}$$

then the candidate subspace must be invariant for a and $a^\dagger$ and so it should be trivial as in the case of two Kraus operators L.

In the sequel, we prove that under condition (9), which is clearly a Hörmander-type iterated commutator condition the QMS is irreducible. Otherwise, we will see that irreducibility does not hold.

It is worth noticing here that a similar condition appears also in bilinear control (see [11], Definition 3.6 (ii) p. 102, *weak ad-condition*) As a matter of fact, if, starting from any initial non-zero vector $\xi_0 \in \mathsf{h}$ with time evolution one can reach a total set of vectors in h varying the control parameter $z \in \mathbb{C}$ in the differential equation $\dot{\xi}_t = G\xi_t + zL\xi_t$, then irreducibility holds.

LEMMA 1 *Suppose* $|v| \neq |u|$. *Then* $\mathrm{Dom}(G_0) = \mathrm{Dom}(N) = \mathrm{Dom}(G)$.

We defer the proof to Appendix B.

PROPOSITION 1 *Suppose that condition (9) holds and, moreover,* $|v| \neq |u|$. *Then the Gaussian QMS with*

$$L \;=\; \overline{v}a + ua^\dagger\,, \qquad H \;=\; \Omega\, a^\dagger a + \frac{\kappa}{2}a^{\dagger 2} + \frac{\overline{\kappa}}{2}a^2 + \frac{\zeta}{2}a^\dagger + \frac{\overline{\zeta}}{2}a$$

is irreducible.

Proof. Knowing that $\mathrm{Dom}(G) = \mathrm{Dom}(N)$ the proof essentially follows the line of that of Theorem 5.

Let $\mathcal{V}$ ($\mathcal{V} \neq \{0\}$) be a subspace of h which is invariant for the operators P_t and $L\,(\mathrm{Dom}(G)\cap\mathcal{V}) \;=\; L\,(\mathrm{Dom}(N)\cap\mathcal{V}) \;\subseteq\; \mathcal{V}$ for $\ell = 1, 2$. Moreover, since $L\,(\mathrm{Dom}(N^m)) \subseteq \mathrm{Dom}(N^{m-1/2})$ for all $m \geq 1/2$ and $G\,(\mathrm{Dom}(N^m)) \subseteq \mathrm{Dom}(N^{m-1})$ for all $m \geq 1$, we have also $[G, L]\left(\mathrm{Dom}(N^{3/2}) \cap \mathcal{V}\right) \subseteq \mathcal{V}$ and $\widetilde{L}\left(\mathrm{Dom}(N^{3/2}) \cap \mathcal{V}\right) \subseteq \mathcal{V}$. However, the commutator $[G, L]$ is a first order polynomial in $a, a^\dagger$, therefore the previous inclusions can be extended to $\mathrm{Dom}(N^{1/2}) \cap \mathcal{V}$.

By the linear independence of L and $\widetilde{L}$, we can now follow the argument of the proof of Theorem 5, with $L_2 = \widetilde{L}$. □

We study separately situations in which (9) does not hold distinguishing three cases.

4.1. The case L of annihilation type

We first consider the case where (9) does not hold and $|v| > |u|$. Act with the unitary squeeze operator $S = e^{(za^{\dagger 2} - \overline{z}a^2)/2}$ ($z \neq 0$, $z = e^{\mathrm{i}\varphi}s$ with $s = |z|$) so that

$$S^* a S \;=\; \cosh(s)\, a + e^{\mathrm{i}\varphi}\sinh(s)\, a^\dagger\,, \quad S^* a^\dagger S \;=\; \cosh(s)\, a^\dagger + e^{-\mathrm{i}\varphi}\sinh(s)\, a\,.$$

Then

$$S^* L S \;=\; \big(\overline{v}\cosh(s) + e^{-\mathrm{i}\varphi}u\sinh(s)\big)\, a + \big(u\cosh(s) + e^{\mathrm{i}\varphi}\overline{v}\sinh(s)\big)\, a^\dagger \quad (10)$$

and, by first choosing a φ such that u and $e^{\mathrm{i}\varphi}\overline{v}$ have the same phase, and an s such that

$$|u|\cosh(s) + |v|\sinh(s) \;=\; 0 \quad \Longleftrightarrow \quad \tanh(s) \;=\; -\frac{|u|}{|v|}$$

we can assume that L is a strictly positive multiple (multiplying L by a phase does not change the GKSL representation) of the annihilation operator, i.e., $u = 0$.

Of course also Ω, κ, ζ change to Ω', κ', ζ'

$$
\begin{aligned}
\Omega' &= \Omega\left(\cosh^2(s)+\sinh^2(s)\right)+2\sinh(s)\cosh(s)\Re(e^{-\mathrm{i}\varphi}\kappa)\\
\kappa' &= \kappa\cosh^2(s)+\overline{\kappa}\,e^{2\mathrm{i}\varphi}\sinh^2(s)+2\Omega\,e^{\mathrm{i}\varphi}\cosh(s)\sinh(s)\\
\zeta' &= \zeta\cosh(s)+\overline{\zeta}e^{\mathrm{i}\varphi}\sinh(s)
\end{aligned}
$$

and condition (9) does not hold if and only if $v\kappa'=0$, i.e., by $v\neq 0$, $\kappa'=0$ and (up to an irrelevant multiple of the identity operator in H)

$$
L=v'a\,,\quad H = \Omega'\,a^\dagger a+\frac{\zeta'}{2}a^\dagger+\frac{\overline{\zeta}'}{2}a\,,\quad G = -\Big(\frac{|v'|^2}{2}+\mathrm{i}\Omega'\Big)a^\dagger a-\frac{\mathrm{i}}{2}(\zeta' a^\dagger+\overline{\zeta}'a)\,,
$$

where $v'=(|v|^2-|u|^2)\cosh(s)/|v|$, up to a phase factor. Dropping the $'$ to simplify the notation, now we apply formula (4) with

$$
Zz = -(|v|^2/2+\mathrm{i}\Omega)z\,,\qquad Cz = |v|^2z\,.
$$

Computing $e^{sZ}z=e^{-(|v|^2/2-\mathrm{i}\Omega)s}z$ and

$$
\int_0^t \Re\left(\overline{e^{sZ}z}\,Ce^{sZ}z\right)\mathrm{d}s = |z|^2\int_0^t |v|^2e^{-s|v|^2}\mathrm{d}s = |z|^2\left(1-e^{-t|v|^2}\right)
$$

$$
\int_0^t \Re\left(\overline{\zeta}e^{sZ}z\right)\mathrm{d}s = \Re\Big(\frac{\overline{\zeta}z}{|v|^2/2-\mathrm{i}\Omega}\left(1-e^{-t(|v|^2/2-\mathrm{i}\Omega)}\right)\Big)\,.
$$

It follows that, for all $g,f\in\mathbb{C}$,

$$
\begin{aligned}
&\lim_{t\to+\infty}\mathrm{tr}(|e(f)\rangle\langle e(g)|\mathcal{T}_t(W(z))\\
&= e^{-|z|^2/2+\mathrm{i}\Re\left(\overline{\zeta}z/(|v|^2/2-\mathrm{i}\Omega)\right)}\lim_{t\to+\infty}\left\langle e(g),W(e^{tZ}z)e(f)\right\rangle\\
&= e^{-|z|^2/2+2\mathrm{i}\Im\left(\overline{\mathrm{i}\zeta}z/(|v|^2-2\mathrm{i}\Omega)\right)}\,e^{\overline{g}f}\,.
\end{aligned}
$$

Noting that, for all $\mu\in\mathbb{C}$

$$
e^{-|\mu|^2}\left\langle e(\mu),W(z)e(\mu)\right\rangle = e^{-|z|^2/2+2\mathrm{i}\Im(\overline{\mu}z)}\,,
$$

defining $\mu=\mathrm{i}\zeta/(|v|^2+2\mathrm{i}\Omega)$ we find

$$
\lim_{t\to+\infty}\mathrm{tr}(|e(f)\rangle\langle e(g)|\mathcal{T}_t(W(z)) = e^{\overline{g}f}e^{-|\mu|^2}\left\langle e(\mu),W(z)e(\mu)\right\rangle.
$$

In particular, $e^{-|\mu|^2}|e(\mu)\rangle\langle e(\mu)|$ is a pure invariant state and the QMS is not irreducible. Moreover, since linear combinations of linear functionals $|e(f)\rangle\langle e(g)|$ are dense in the Banach space of trace class operators by totality of exponential vectors, that the above identity also proves that any initial state converges in trace norm to this pure invariant state.

PROPOSITION 2 *The Gaussian QMS with GKSL generator with only one Kraus operator* $L = \overline{v}a + ua^\dagger$, $|v| > |u|$ *and Hamiltonian* H *as in (3) is irreducible if and only if condition (9) holds. If it is not irreducible, it has a unique invariant state* $e^{-|\mu|^2}|e(\mu)\rangle\langle e(\mu)|$ *(pure) and all initial state converges to it in trace norm.*

Clearly, after our squeeze transformation $\mu = \mathrm{i}\zeta'/(|v'|^2 + 2\mathrm{i}\Omega')$.

4.2. The case L of creation type

We consider the case where (9) does not hold and $|v| < |u|$. First choosing a ϕ such that u and $e^{\mathrm{i}\phi}\overline{v}$ have the same phase, and then θ such that $\tanh(\theta) = |v|/|u|$ in (10) we can assume $v = 0$ and L multiple of the creation operator. Parameters Ω, κ, ζ are transformed to Ω', κ', ζ' and (9) does not hold if and only if $\kappa' = 0$. In this way the given QMS is transformed to the unitarily equivalent QMS generated by

$$L = u'a^\dagger\,, \quad H = \Omega'\, a^\dagger a + \frac{\zeta'}{2}a^\dagger + \frac{\overline{\zeta}'}{2}a\,, \quad G = -\Big(\frac{|u'|^2}{2} + \mathrm{i}\Omega'\Big)aa^\dagger - \frac{\mathrm{i}}{2}(\zeta' a^\dagger + \overline{\zeta}' a),$$

where $u' = (|u|^2 - |v|^2)\cosh(\theta)/|u|$ up to a phase factor. In the sequel we drop the $'$ to simplify the notation. Let $\mathcal{V}$ be the range of a nonzero subharmonic projection p. Since, by Lemma 1 the operators G and N have the same domain, by Theorem 3 we have $G(\mathrm{Dom}(N)\cap\mathcal{V}) \subseteq \mathcal{V}$ $L(\mathrm{Dom}(N)\cap\mathcal{V}) \subseteq \mathcal{V}$. Adding to G a suitable multiple of L we find the operator

$$\begin{aligned} \widetilde{G} &= -\Big(\frac{|u|^2}{2} + \mathrm{i}\Omega\Big)\Big(a\,a^\dagger + \overline{\eta}a + \eta a^\dagger + |\eta|^2\mathbb{1}\Big) \\ &= -\Big(\frac{|u|^2}{2} + \mathrm{i}\Omega\Big)W(\eta)^* a\,a^\dagger W(\eta)\,, \end{aligned}$$

where $\eta = \mathrm{i}\zeta/(|u|^2 - 2\mathrm{i}\Omega)$ such that $\widetilde{G}(\mathrm{Dom}(N)\cap\mathcal{V}) \subseteq \mathcal{V}$. This property, together with $L(\mathrm{Dom}(N)\cap\mathcal{V}) \subseteq \mathcal{V}$, is clearly equivalent to G, L invariance.

Let $w \in \mathcal{V}$ with expansion $w = \sum_{k\geq k_0} w_k\, W(-\eta)e_k$ where k_0 is the minimum k for which $w_k \neq 0$. Since $\widetilde{G}$ is a multiple of the number operator with strictly negative real part, arguing as in the last part of the proof of Theorem 5, we can show that $W(-\eta)e_{k_0} \in \mathcal{V}$. As a consequence, by the commutation $a^\dagger W(-\eta) = W(-\eta)(a^\dagger - \overline{\eta}\mathbb{1})$,

$$\begin{aligned} LW(-\eta)e_{k_0} &= u\,W(-\eta)(a^\dagger - \overline{\eta}\mathbb{1})e_{k_0} \\ &= u\sqrt{k_0+1}\,W(-\eta)e_{k_0+1} - u\,\overline{\eta}W(-\eta)e_{k_0} \;\in\; \mathcal{V}\,. \end{aligned}$$

Applying L we can show inductively that, for all $k_0 \geq 0$, the linear space generated by vectors $W(-\eta)e_k$ with $k \geq k_0$ is an invariant subspace determining a subharmonic projection and, in this case, the QMS associated with G, L is not irreducible.

4.3. The case L quadrature (selfadjoint)

We consider the case where (9) does not hold and $|v| = |u|$ so that, $v = re^{\mathrm{i}\alpha}$, $u = re^{\mathrm{i}\alpha'}$ with $r > 0$ and

$$L \;=\; re^{-\mathrm{i}\alpha}a + re^{\mathrm{i}\alpha'}a^\dagger \;=\; re^{\mathrm{i}(\alpha'-\alpha)/2}\left(e^{-\mathrm{i}(\alpha'+\alpha)/2}a + re^{\mathrm{i}(\alpha'+\alpha)/2}a^\dagger\right).$$

Therefore, multiplying L by a phase $e^{-\mathrm{i}(\alpha'-\alpha)/2}$ and putting $\alpha' + \alpha = 2\theta$ we get the selfadjoint operator L

$$L \;=\; r(e^{-\mathrm{i}\theta}a + e^{\mathrm{i}\theta}a^\dagger)$$

which is a positive multiple of a quadrature. We could also reduce ourselves to the case where θ is zero by applying a unitary transformation $e^{\mathrm{i}\theta N}$ on $\Gamma(\mathbb{C})$, however we prefer to keep the parameter θ to highlight the relationship between the phase θ in the operator L and another phase of the coefficients κ of the Hamiltonian H.

Indeed, in a similar way, putting $\kappa = |\kappa|e^{2\mathrm{i}\phi}$ we can write

$$H \;=\; \Omega\, a^\dagger a + \frac{|\kappa|}{2}\left(e^{2\mathrm{i}\phi}a^{\dagger 2} + e^{-2\mathrm{i}\phi}a^2\right) + \left(\overline{\zeta}a + \zeta a^\dagger\right).$$

Considering quadratures with angle θ given by the selfadjoint

$$q_\theta \;=\; \left(e^{-\mathrm{i}\theta}a + e^{\mathrm{i}\theta}a^\dagger\right)/\sqrt{2}$$

and noting that

$$a^\dagger \;=\; e^{-\mathrm{i}\theta}\left(q_\theta - \mathrm{i}q_{\theta+\pi/2}\right)/\sqrt{2} \qquad a \;=\; e^{\mathrm{i}\theta}\left(q_\theta + \mathrm{i}q_{\theta+\pi/2}\right)/\sqrt{2} \tag{11}$$

$$a^\dagger a + aa^\dagger \;=\; q_\theta^2 + q_{\theta+\pi/2}^2 \tag{12}$$

we can write H as

$$\begin{aligned} H \;=\;& \frac{\Omega + |\kappa|\cos(2(\phi-\theta))}{2}\, q_\theta^2 + \frac{\Omega - |\kappa|\cos(2(\phi-\theta))}{2}\, q_{\theta+\pi/2}^2 \\ +\;& \frac{|\kappa|}{2}\sin(2(\phi-\theta))\left(q_\theta q_{\theta+\pi/2} + q_{\theta+\pi/2}q_\theta\right) + (\overline{\zeta}a + \zeta a^\dagger) - \frac{\Omega}{2}\mathbb{1}. \end{aligned} \tag{13}$$

We can immediately see that (9) does not hold if and only if

$$\Omega \;=\; |\kappa|\cos(2(\phi-\theta)) \tag{14}$$

the quadratic term $q_{\theta+\pi/2}^2$ in H vanishes and the Abelian algebra generated by the position operator q_θ is invariant. Indeed, for all smooth function

$f : \mathbb{R} \to \mathbb{C}$, we have $[L, f(q_\theta)] = [L^*, f(q_\theta)] = 0$ and by the identities

$$\begin{aligned}
\left[q_{\theta+\pi/2}, f(q_\theta)\right] &= \left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}q_\theta}, f(q_\theta)\right] = \mathrm{i}f'(q_\theta) \\
[a, f(q_\theta)] &= \frac{e^{\mathrm{i}\theta}}{\sqrt{2}}\left[q_\theta - \mathrm{i}q_{\theta+\pi/2}, f(q_\theta)\right] = \frac{e^{\mathrm{i}\theta}}{\sqrt{2}}\left[\frac{d}{dq_\theta}, f(q_\theta)\right] \\
&= \frac{e^{\mathrm{i}\theta}}{\sqrt{2}} f'(q_\theta) \\
[a^\dagger, f(q_\theta)] &= \frac{e^{-\mathrm{i}\theta}}{\sqrt{2}}\left[q_\theta + \mathrm{i}q_{\theta+\pi/2}, f(q_\theta)\right] = \frac{e^{-\mathrm{i}\theta}}{\sqrt{2}}\left[-\frac{d}{dq_\theta}, f(q_\theta)\right] \\
&= \frac{-e^{-\mathrm{i}\theta}}{\sqrt{2}} f'(q_\theta)\,.
\end{aligned}$$

As a result, we find

$$\mathcal{L}(f(q_\theta)) = \mathrm{i}\,[H, f(q_\theta)] = \left(\Im(\zeta e^{-\mathrm{i}\theta})/\sqrt{2} - |\kappa|\sin(2(\theta-\phi))q_\theta\right) f'(q_\theta)\,.$$

Note that, if the quadratic term $q^2_{\theta+\pi/2}$ in H does not vanish, then we can not get the same conclusion.

This is the generator of a deterministic translation process with drift (in the generic case where $|\kappa|\sin(2(\theta-\phi)) \neq 0$) towards the point $x_\infty := \Im(\zeta e^{-\mathrm{i}\theta})/(\sqrt{2}|\kappa|\sin(2(\theta-\phi)))$ (Fig. 1 below).

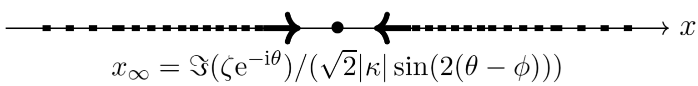


Fig. 1: Deterministic translation process on the algebra generated by q_θ.

The invariant density of the classical process is clearly δ_{x_∞} which does not induce a faithful normal state on $\mathcal{B}(\mathsf{h})$. However this insight turns out to be useful to demonstrate that the QMS we are considering in this subsection is not irreducible if (14) holds. For all $c > 0$ consider the projection

$$x \longmapsto 1_{[x_\infty - c, x_\infty + c]}(x)$$

which is a candidate subharmonic projection because the classical process, starting from a point in the interval $[x_\infty - c, x_\infty + c]$ does not exit for all positive times.

To prove that this projection is indeed subharmonic, consider mollifier φ, namely a C^∞ function $\varphi : \mathbb{R} \to \mathbb{R}_+$ with support in the interval $[-1, 1]$,

$\int_{\mathbb{R}} \varphi(x)\mathrm{d}x = 1$ and $\lim_{\epsilon\to 0} \varphi_\epsilon(x) = \lim_{\epsilon\to 0} \epsilon^{-1}\varphi(x/\epsilon) = \delta_0$ and, for all $\epsilon < c$ define

$$f_\epsilon(x) \;=\; \int_{-\infty}^{x} \left(\varphi_\epsilon(y-(x_\infty - c)) - \varphi_\epsilon(y-(x_\infty + c))\right) \mathrm{d}y\,.$$

Note that, since $\int_{\mathbb{R}} \varphi_\epsilon(x)\mathrm{d}x = 1$ for all $\epsilon > 0$ we have $f_\epsilon(x) = 0$ for $|x - x_\infty| > c+\epsilon$, $f(x) = 1$ for $|x - x_\infty| \le c-\epsilon$ and $f_\epsilon'(x) \ge 0$ for $x_\infty - c - \epsilon < x < x_\infty - c + \epsilon$, $f_\epsilon'(x) \le 0$ for $x_\infty + c - \epsilon < x < x_\infty + c + \epsilon$. It follows that the multiplication operator by $f_\epsilon(q_\theta)$, which belongs to the domain of the Lindbladian $\mathcal{L}$ because $\mathcal{L}(f_\epsilon(q_\theta))$ is bounded satisfies $\mathcal{L}(f_\epsilon(q_\theta)) \ge 0$ and so

$$\mathcal{T}_t(f_\epsilon(q_\theta)) \;\ge\; f_\epsilon(q_\theta)$$

for all $t \ge 0$. Taking the limit as ϵ goes to 0, f_ϵ converges to the projection $1_{[x_\infty - c, x_\infty + c]}$ in L^2 and almost surely, therefore $\mathcal{T}_t(1_{[x_\infty - c, x_\infty + c]}) \ge 1_{[x_\infty - c, x_\infty + c]}$ for all $t \ge 0$ and the QMS is *not* irreducible.

A similar argument applies in the case where $\sin(2(\theta-\phi)) = 0$ and $x_\infty = +\infty$ (resp. $x_\infty = -\infty$) if $\Im(\zeta \mathrm{e}^{-\mathrm{i}\theta}) > 0$ (resp. $\Im(\zeta \mathrm{e}^{-\mathrm{i}\theta}) < 0$) with projections of the form $1_{[c,+\infty[}$ (resp. $1_{]-\infty,c]}$).

We now consider the case where $|u| = |v|$ and condition (9) holds, namely $\Omega \neq |\kappa|\cos(2(\theta-\phi))$ and show that the QMS is irreducible. To this end we need the following result also showing that irreducibility is equivalent to coercivity of $G_0^2 + H^2 + g_l^2 \mathbb{1}$, for some constant g_l^2, with respect to the graph norm of the number operator $N = (q_\theta^2 + q_{\theta+\pi/2}^2 - 1)/2$.

Intuitively, looking at formula (13), one sees that the coefficient of $q_{\theta+\pi/2}^2$ is non-zero if $\Omega \neq |\kappa|\cos(2(\theta-\phi))$. Therefore, computing H^2, the coefficient of $q_{\theta+\pi/2}^4$ is non-zero. The coefficient of q_θ^4 may vanish but one gets an additional term r^4 by addition of G_0^2 and strict positivity of leading terms is restored.

THEOREM 6 *If condition (9) holds, namely $\Omega \neq |\kappa|\cos(2(\phi-\theta))$, there exist constants $g^2 > 0$, $g_l^2 \ge 0$ such that*

$$G_0^2 + H^2 \;\ge\; g^2 (q_\theta^2 + q_{\theta+\pi/2}^2)^2 - g_l^2 \mathbb{1}\,. \tag{15}$$

In particular $\mathrm{Dom}(G) = \mathrm{Dom}(N)$.

Proof. In this proof only, to reduce the clutter of the notation, we denote q_θ by q, $q_{\theta+\pi/2}$ by p, $c := \cos(2(\phi-\theta))$, $s := \sin(2(\phi-\theta))$ and by $\{\cdot,\cdot\}$ the anticommutator.

As a first step note that, once we show that $G_0^2 + H^2 \ge g_0^2(q_\theta^2 + q_{\theta+\pi/2}^2)^2 +$ l.o.t. for some constant $g_0^2 > 0$ then, reducing the constant g_0 if necessary, we can get the conclusion. Indeed, if the lower order term is, for instance,

$\{a, q^2 + p^2\}$ for all $\xi \in \mathrm{Dom}(N^2)$ by the Schwartz and Young inequalities, we have

$$\begin{aligned}\left\langle \xi, \{a, q^2+p^2\}\xi \right\rangle &= \langle a^\dagger \xi, (q^2+p^2)\xi\rangle + \left\langle (q^2+p^2)\xi, a\,\xi \right\rangle \\ &\geq -\|a^\dagger\,\xi\| \cdot \left\|(q^2+p^2)\xi\right\| - \|a\,\xi\| \cdot \left\|(q^2+p^2)\xi\right\| \\ &\geq -\epsilon \left\|(q^2+p^2)\xi\right\|^2 - \epsilon^{-1}(\|a\,\xi\|^2 + \|a^\dagger\,\xi\|^2) \\ &= -\epsilon \left\langle \xi, (q^2+p^2)^2\xi \right\rangle - \epsilon^{-1}\left\langle \xi, (q^2+p^2)\xi \right\rangle\end{aligned}$$

for all $\epsilon > 0$. Now, again by the Schwartz and Young inequalities we have also

$$\begin{aligned}-\epsilon^{-1}\left\langle \xi, (q^2+p^2)\xi \right\rangle &\geq -\epsilon^{-1}\|\xi\| \cdot \left\|(q^2+p^2)\xi\right\| \\ &\geq -\epsilon \left\|(q^2+p^2)\xi\right\|^2 - \epsilon^{-3}\|\xi\|^2 .\end{aligned}$$

Therefore we find the inequality

$$\left\langle \xi, \{a, q^2+p^2\}\xi \right\rangle \geq -2\epsilon \left\langle \xi, (q^2+p^2)^2\xi \right\rangle - \epsilon^{-3}\|\xi\|^2$$

and, choosing ϵ small enough, we can reduce the constant g^2 in (15), increase g_l^2 and get the claimed inequality. We can proceed in a similar way if there are more lower order terms.

It is now clear that we can assume that $G_0^2 + H^2$ is a fourth order *homogenous* polynomial in p, q, or, in an equivalent way, we can proceed as if H had no terms of order 1 or 0. In this case the square of $2H$ is

$$\begin{aligned}(2H)^2 &= (\Omega + |\kappa|c)^2\, q^4 + (\Omega - |\kappa|c)^2\, p^4 + (\Omega + |\kappa|c)\,(\Omega - |\kappa|c)\,\{q^2, p^2\} \\ &\quad + |\kappa|^2 s^2 \{q,p\}^2 + (\Omega + |\kappa|c)\,|\kappa|s\,\left\{q^2, \{q,p\}\right\} \\ &\quad + \ (\Omega - |\kappa|c)\,|\kappa|s\,\left\{p^2, \{q,p\}\right\}\end{aligned}$$

and write $(2H)^2$ as

$$[q^2, \{q,p\}, p^2] \begin{bmatrix} (\Omega+|\kappa|c)^2 & (\Omega+|\kappa|c)\,|\kappa|s & (\Omega+|\kappa|c)\,(\Omega-|\kappa|c) \\ (\Omega+|\kappa|c)\,|\kappa|s & |\kappa|^2 s^2 & (\Omega-|\kappa|c)\,|\kappa|s \\ (\Omega+|\kappa|c)\,(\Omega-|\kappa|c) & (\Omega-|\kappa|c)\,|\kappa|s & (\Omega-|\kappa|c)^2 \end{bmatrix} \begin{bmatrix} q^2 \\ \{q,p\} \\ p^2 \end{bmatrix}.$$

We now apply Lemma 8 Appendix C on a 3×3 matrix as above with $\lambda = \Omega - |\kappa|c, \mu = \Omega + |\kappa|c, x = |\kappa|s$. Since $L = \sqrt{2}rq$ and $G_0 = -r^2q^2$, the operator $(2G_0)^2 + (2H)^2$ is associated with a 3×3 matrix as in Lemma 8 therefore is bigger than (r^4 becomes $4r^4$)

$$\epsilon \left(4r^4q^4 - \{\,p,q\,\}^2/2 + \{\,p^2, q^2\,\} + \lambda^2 w\, q^4\right) + \mathrm{l.o.t.}$$

Note that $\{\,p^2, q^2\,\} - \{\,p,q\,\}^2/2 = -(3/2)\mathbb{1}$ and $\{\,p^2, q^2\,\} \leq p^4 + q^4$ which implies

$$\begin{aligned}4(G_0^2 + H^2) &\geq \epsilon\left(4r^4q^4 + \lambda^2 w\, q^4\right) + \mathrm{l.o.t.} \\ &\geq \epsilon \min\{2r^4, \lambda^2 w/2\}\left(q^4 + \{\,p^2, q^2\,\} + p^4\right) + \mathrm{l.o.t.} \\ &= \epsilon \min\{2r^4, \lambda^2 w/2\}\left(q^2 + p^2\right)^2 + \mathrm{l.o.t.}\end{aligned}$$

The above inequality together with (33) implies existence of constants, $g, g' > 0$ such that

$$\|N\xi\|^2 \ \leq \ g\|G\xi\|^2 + g'\|\xi\|^2$$

for all ξ finite linear combination of vectors e_n of the c.o.n.b. Therefore $\mathrm{Dom}(G) \subseteq \mathrm{Dom}(N)$. The other inclusion is trivial and the proof is complete. □

Let $\mathcal{V}$ be the range of a subharmonic projections. By the previous arguments based on $a, a^\dagger$ invariance of $\mathcal{V} \cap \mathrm{Dom}(N)$ and $\mathrm{Dom}(G) = \mathrm{Dom}(N)$ as in the proof of Theorem 5 we can now prove the following

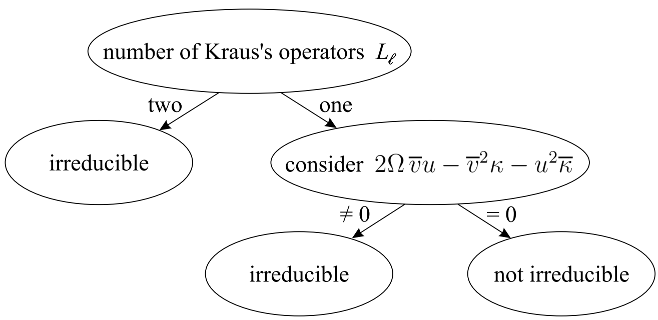
THEOREM 7 *Let $\mathcal{T}$ be the QMS with generator in a generalized GKSL form associated with a single Kraus operator $L = \overline{v}a + ua^\dagger$ and H as in (3). The following are equivalent:*

(1) *Operators L and $[H, L]$ are linearly independent, i.e., $2\Omega\,\overline{v}u \neq \overline{v}^2\kappa + u^2\overline{\kappa}$,*

(2) *$\mathcal{T}$ is irreducible.*

Proof. (1) ⇒ (2) If $|u| \neq |v|$, the conclusion follows from Proposition 1. If $|u| = |v|$, we know from Theorem 6 that $\mathrm{Dom}(G) = \mathrm{Dom}(N)$ therefore the proof of Proposition 1 goes through again and shows that $\mathcal{T}$ is irreducible.
(2) ⇒ (1) We showed, in Sect. 4.1 for $|u| < |v|$, in Sect. 4.2 for $|u| > |v|$, and in Sect. 4.3 for $|u| = |v|$, that if condition (1) does not hold then the QMS $\mathcal{T}$ is not irreducible. □

Solution to the irreducibility problem is summarized by the following decision tree.



5. Gaussian States

As a preliminary to the study of invariant states, in this section, we recall some basic properties of Gaussian states of a one-dimensional CCR algebra.

DEFINITION 2 A density matrix ρ is called a quantum gaussian state if there exist $\omega \in \mathbb{C}$ and a real linear, symmetric, invertible operator S such that

$$\hat{\rho}(z) \;=\; \exp\Big(-\frac{1}{2}\Re(\overline{z}\, Sz) - \mathrm{i}\Re(\overline{\omega} z)\Big) \quad \forall\, z \in \mathbb{C}\,. \tag{16}$$

In that case ω is said to be the *mean vector* and S the *covariance operator* and we will denote it also with $\rho_{(\omega,S)}$.

This notation is well posed since there is a bijection between density matrices and characteristic functions.

Let S be a real linear operator on $\mathbb{C}$ and $z = x + \mathrm{i}y \in \mathbb{C}$. In the following it will be useful to identify them with a real linear operator $\mathbf{S}$ acting on $\mathbb{R}^2$ and a vector $\mathbf{z}$ in $\mathbb{R}^2$ that will be denoted with characters in boldface for the sake of clarity. Namely, $\boldsymbol{z} = (x, y)$ and, if $Sz = s_1 z + s_2\overline{z}$ for every $z \in \mathbb{C}$, we have

$$\boldsymbol{S}\boldsymbol{z} \;=\; \begin{bmatrix} \Re s_1 + \Re s_2 & \Im s_2 - \Im s_1 \\ \Im s_1 + \Im s_2 & \Re s_1 - \Re s_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Vice versa, given a linear operator on $\mathbb{R}^2$

$$\boldsymbol{S} \;=\; \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

we can induce a real linear operator S on $\mathbb{C}$ via

$$Sz = \left(\frac{S_{11}+S_{22}}{2} + \mathrm{i}\frac{S_{21}-S_{12}}{2}\right) z + \left(\frac{S_{11}-S_{22}}{2} + \mathrm{i}\frac{S_{12}+S_{21}}{2}\right)\overline{z}.$$

In the following we will also use J as the linear linear operator corresponding to the multiplication by $-\mathrm{i}$, namely

$$Jz \;=\; -\mathrm{i}z\,, \quad \boldsymbol{J} \;=\; \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Eventually we will denote the adjoint of S with respect to the real scalar product $(z, w) \to \Re(\overline{z}w)$ with S^T. This operator is explicitly given by

$$S^T z \;=\; \overline{s_1} z + s_2 \overline{z}\,,$$

while $\boldsymbol{S}^T$ is given by the usual matrix transposition of $\boldsymbol{S}$.

Remark 2 For a generic real linear, symmetric, invertible operator S to be a suitable covariance operator of a gaussian state it also needs to satisfy

$$\boldsymbol{S} - \mathrm{i}\boldsymbol{J} \;\geq\; 0\,, \tag{17}$$

where $\boldsymbol{S}$ and $\boldsymbol{J}$ are now intended as complex linear operator on $\mathbb{C}^2$ (see [24, Theorem 3.1] with a little warning: J there is the multiplication by i). Therefore positivity is evaluated with respect to the usual complex inner product.

LEMMA 2 *The real linear operators* C *and* Z *given by (5), (6) satisfy*

$$\boldsymbol{C} + \mathrm{i}\left(\boldsymbol{Z}^T \boldsymbol{J} + \boldsymbol{J}\boldsymbol{Z}\right) \;\geq\; 0\,, \quad \boldsymbol{C} \geq 0\,,$$

where the first inequality is intended with respect to the complex scalar product on $\mathbb{C}^2$*. Moreover, the first inequality holds strictly if and only if there are exactly two linear independent Kraus operator* L_1, L_2*. The second one is strict if and only if the parameter*

$$\gamma \;=\; \frac{1}{2}\sum_{\ell=1}^{2}\left(|v_\ell|^2 - |u_\ell|^2\right)$$

is non-zero.

Proof. Let us begin by observing that

$$\boldsymbol{C} \;=\; \sum_\ell \begin{bmatrix} |u_\ell + \overline{v}_\ell|^2 & 2\Im(v_\ell u_\ell) \\ 2\Im(v_\ell u_\ell) & |u_\ell - \overline{v}_\ell|^2 \end{bmatrix}, \quad \boldsymbol{Z} \;=\; \begin{bmatrix} -\gamma - \Im(\kappa) & \Re(\kappa) - \Omega \\ \Re(\kappa) + \Omega & -\gamma + \Im(\kappa) \end{bmatrix}. \tag{18}$$

Now a straightforward computation leads to

$$\boldsymbol{C} + \mathrm{i}\left(\boldsymbol{Z}^T \boldsymbol{J} + \boldsymbol{J}\boldsymbol{Z}\right) \;=\; 2\begin{bmatrix} \sum_\ell \frac{|u_\ell + \overline{v}_\ell|^2}{2} & -\mathrm{i}\gamma + \sum_\ell \Im(v_\ell u_\ell) \\ \mathrm{i}\gamma + \sum_\ell \Im(v_\ell u_\ell) & \sum_\ell \frac{|u_\ell - \overline{v}_\ell|^2}{2} \end{bmatrix} \;=:\; 2\,\boldsymbol{Q}$$

hence positivity of $\boldsymbol{C} + \mathrm{i}\left(\boldsymbol{Z}^T \boldsymbol{J} + \boldsymbol{J}\boldsymbol{Z}\right)$ is equivalent to positivity of $\boldsymbol{Q}$. One has $\mathrm{tr}(\boldsymbol{Q}) > 0$, while

$$\det \boldsymbol{Q} \;=\; \frac{1}{4}\Big(\sum_\ell |u_\ell + \overline{v}_\ell|^2\Big)\Big(\sum_\ell |u_\ell - \overline{v}_\ell|^2\Big) - \Big(\sum_\ell \Im(u_\ell v_\ell)\Big)^2 - \gamma^2\,.$$

Now we can use $|u_\ell \pm \overline{v}_\ell|^2 = |u_\ell|^2 + |v_\ell|^2 \pm 2\Re(u_\ell v_\ell)$ in order to obtain

$$\det \boldsymbol{Q} \;=\; \Big(\sum_\ell |u_\ell|^2\Big)\Big(\sum_\ell |v_\ell|^2\Big) - \Big|\sum_\ell u_\ell v_\ell\Big|^2$$

which is positive by the Cauchy-Schwarz inequality. In the case where there is only a Kraus operator L_1 clearly $\det(\boldsymbol{Q}) = 0$. Conversely, if $\det(\boldsymbol{Q}) = 0$ then the Cauchy-Schwarz inequality becomes an equality, therefore we can find $\lambda \in \mathbb{C}$ such that $u_\ell = \lambda v_\ell$ for every $\ell = 1, 2$ which contradicts linear independence of L_1 and L_2. The analysis of the first inequality is now complete.

For the second one observe that $\mathrm{tr}(\boldsymbol{C}) \geq 0$ and, with similar computations,

$$\begin{aligned}
\det(\boldsymbol{C}) &= \Big(\sum_\ell \left(|u_\ell|^2 + |v_\ell|^2\right)\Big)^2 - 4\Big|\sum_\ell u_\ell v_\ell\Big|^2 \\
&\geq \Big(\sum_\ell \left(|u_\ell|^2 + |v_\ell|^2\right)\Big)^2 - 4\Big(\sum_\ell |u_\ell|^2\Big)\Big(\sum_\ell |v_\ell|^2\Big) \\
&= \Big(\sum_\ell \left(|u_\ell|^2 - |v_\ell|^2\right)\Big)^2 = 4\gamma^2 \geq 0\,.
\end{aligned}$$

This completes the proof. □

6. Invariant States

In this section we characterize Gaussian QMS with normal invariant states in terms of the parameters in the model. We begin by the explicit formula for the action of the predual semigroup on Gaussian states.

PROPOSITION 3 *Let $(\mathcal{T}_t)_{t\geq 0}$ be the quantum Markov Semigroup with GKSL generator associated with H, L_1, L_2 as in (2), (3) and let $(\mathcal{T}_{*t})_{t\geq 0}$ be its predual semigroup. If $\rho = \rho_{(\omega_0, S_0)}$ is a gaussian state then $\rho_t := \mathcal{T}_{*t}(\rho)$ is still a Gaussian state for every $t \geq 0$ with mean vector ω_t and covariance operator S_t given by*

$$\omega_t = e^{tZ^T}\omega_0 - \int_0^t e^{sZ^T}\zeta \mathrm{d}s \tag{19}$$

$$S_t = e^{tZ^T} S_0 e^{tZ} + \int_0^t e^{sZ^T} C e^{sZ} \mathrm{d}s\,. \tag{20}$$

Proof. Applying the explicit formula (4) of Theorem 2 we can write

$$\begin{aligned}
\hat{\rho}_t(z) &= \mathrm{tr}(\rho \mathcal{T}_t(W(z))) \\
&= \exp\left(-\frac{1}{2}\Re\left\langle z, \int_0^t e^{sZ^T} C e^{sZ} z \mathrm{d}s\right\rangle - \frac{1}{2}\Re\left\langle z, e^{tZ^T} S_0 e^{tZ} z\right\rangle\right) \\
&\times \exp\left(\mathrm{i}\Re\left\langle \int_0^t e^{sZ^T}\zeta \mathrm{d}s, z\right\rangle - \mathrm{i}\Re\left\langle e^{tZ^T}\omega_0, z\right\rangle\right).
\end{aligned}$$

Comparing the previous equation with (16) we find (20) and (19). Now for S_t to be a suitable covariance matrix it should hold $\boldsymbol{S}_t - \mathrm{i}J \geq 0$. Indeed, using $\boldsymbol{S}_0 - \mathrm{i}\boldsymbol{J} \geq 0$ and Lemma 2, one gets

$$\begin{aligned}\boldsymbol{S}_t - \mathrm{i}\boldsymbol{J} &\geq \int_0^t e^{s\boldsymbol{Z}^T}\boldsymbol{C}e^{s\boldsymbol{Z}}ds + e^{t\boldsymbol{Z}^T}\mathrm{i}\boldsymbol{J}e^{t\boldsymbol{Z}} - \mathrm{i}\boldsymbol{J} \\ &= \int_0^t e^{s\boldsymbol{Z}^T}\left(\boldsymbol{C} + \mathrm{i}\left(\boldsymbol{Z}^T\boldsymbol{J} + \boldsymbol{J}\boldsymbol{Z}\right)\right)e^{s\boldsymbol{Z}}\mathrm{d}s \;\geq\; 0\,.\end{aligned}$$

Note that all the operators in the previous inequality were considered as complex linear and therefore commutation with i was legit. □

Remark 3 One could extend Proposition 3 proving that QMSs with generalized GKSL generator (1) with Hamiltonian H given by (3) and two, one or none operator L_ℓ linear in $a, a^\dagger$ as (2) form the most general class of weakly* continuous semigroups of completely positive, identity preserving maps on $\mathcal{B}(\Gamma(\mathbb{C}))$ that preserve Gaussian states. We omit the proof because, albeit interesting, on one hand it would be too long, and, on the other hand, it is just a slight extension of existing results. Here we limit ourselves to mention some bibliographic references. The proof can be done in two steps by characterizing, first completely positive maps preserving Gaussian states, then semigroups composed by such maps. The first step is quite lengthy because it involves the extension of Theorem 4.5 [15], proved for automorphisms, to general completely positive maps. This proof occupies the major part of the paper, accounting for Lemmas and accessory results. An extension of Theorem 4.5 would not be too difficult but lengthy. Indeed the authors of [8], at the very end of the paper, claim such an extension is possible, without spelling out the details.

We now turn our attention to finding Gaussian invariant states.

THEOREM 8 *Let $(\mathcal{T}_t)_{t\geq 0}$ be the QMS with GKSL generator associated with H, L_1, L_2 as in (2), (3) or with H and a single Kraus operator. If $\gamma > 0$ and $\gamma^2 + \Omega^2 - |\kappa|^2 > 0$ the Gaussian state $\rho = \rho_{(\omega,\boldsymbol{S})}$ with*

$$\omega \;=\; (Z^T)^{-1}\zeta \;=\; \frac{(-\gamma + \mathrm{i}\Omega)\zeta - \mathrm{i}\kappa\overline{\zeta}}{\gamma^2 + \Omega^2 - |\kappa|^2}\,, \qquad S \;=\; \int_0^\infty e^{sZ^T}Ce^{sZ}\mathrm{d}s \tag{21}$$

is the unique normal invariant state for the semigroup. Moreover, for all initial state ρ_0

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathcal{T}_{*s}(\rho_0)\mathrm{d}s \;=\; \rho$$

in trace norm.

Proof. First note that, since $\gamma^2+\Omega^2-\kappa^2>0$, the matrix $\boldsymbol{Z}$ in (18) has eigenvalues with *strictly negative* real part, therefore the integral in (21) is well-defined.

We now check that ρ is an invariant state. Proposition 3 implies that $\rho_t=\mathcal{T}_{*t}(\rho)$ is still a Gaussian state with mean vector and covariance matrix given by equations (19) and (20). The state ρ is invariant if and only if $\omega_t=\omega$ and $\boldsymbol{S}_t=\boldsymbol{S}$ for every $t\geq 0$ that means

$$\int_0^t e^{s\boldsymbol{Z}^T}(\boldsymbol{Z}^T\boldsymbol{\omega}-\boldsymbol{\zeta})\mathrm{d}s \;=\; 0\,, \quad \int_0^t e^{s\boldsymbol{Z}^T}(\boldsymbol{C}+\boldsymbol{Z}^T\boldsymbol{S}+\boldsymbol{S}\boldsymbol{Z})e^{s\boldsymbol{Z}}\mathrm{d}s \;=\; 0$$

for all $t\geq 0$. Since both $e^{s\boldsymbol{Z}^T}$ and $e^{s\boldsymbol{Z}}$ are invertible, the invariance of ρ is equivalent to

$$\boldsymbol{\zeta} \;=\; \boldsymbol{Z}^T\boldsymbol{\omega}\,, \quad \boldsymbol{Z}^T\boldsymbol{S}+\boldsymbol{S}\boldsymbol{Z} \;=\; -\boldsymbol{C}\,. \tag{22}$$

Conditions on the parameters of the semigroup imply the existence of a pair $(\boldsymbol{\omega},\boldsymbol{S})$ satisfying (22). Indeed $\gamma^2+\Omega^2-|\kappa|^2\neq 0$ implies invertibility of $\boldsymbol{Z}^T$, which leads to $\boldsymbol{\omega}=(\boldsymbol{Z}^T)^{-1}\boldsymbol{\zeta}$. Furthermore

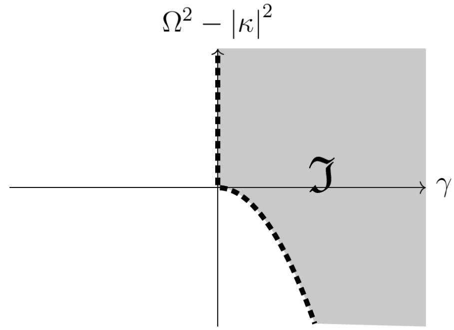
$$\begin{aligned}
\boldsymbol{Z}^T\boldsymbol{S}+\boldsymbol{S}\boldsymbol{Z} \;&=\; \int_0^\infty \left(\boldsymbol{Z}^T e^{s\boldsymbol{Z}^T}\boldsymbol{C}e^{s\boldsymbol{Z}}+e^{s\boldsymbol{Z}^T}\boldsymbol{C}e^{s\boldsymbol{Z}}\boldsymbol{Z}\right)\mathrm{d}s \\
&=\; \int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}s}e^{s\boldsymbol{Z}^T}\boldsymbol{C}e^{s\boldsymbol{Z}}\right)\mathrm{d}s \\
&=\; \left[e^{s\boldsymbol{Z}^T}\boldsymbol{C}e^{s\boldsymbol{Z}}\right]_0^\infty \;=\; -\boldsymbol{C}\,.
\end{aligned}$$

Moreover we can show as in the proof of Proposition 3 that S is a suitable covariance matrix by noting that

$$\boldsymbol{S}-\mathrm{i}\boldsymbol{J} \;=\; \int_0^\infty e^{s\boldsymbol{Z}^T}(\boldsymbol{C}+\mathrm{i}\left(\boldsymbol{Z}^T\boldsymbol{J}+\boldsymbol{J}\boldsymbol{Z}\right))e^{s\boldsymbol{Z}}\mathrm{d}s\,,$$

which exists, since $\boldsymbol{Z}$ has only eigenvalues with negative real part, and is positive thanks to Lemma 2.

Uniqueness follows from irreducibility by standard results on QMS with faithful normal invariant states (see, e.g., [16] Theorem 1 and Lemma 1), otherwise it follows from Proposition 2 since $\gamma>0$. Convergence towards the invariant state follows in a similar way either from known results on QMS with faithful normal invariant states (see, e.g., [17, Theorem 2.1]), or from Proposition 2 since $\gamma>0$. □



$$\mathfrak{I} = \left\{ (\gamma, \Omega^2 - |\kappa|^2) \mid \gamma > 0,\ \gamma^2 + \Omega^2 - |\kappa|^2 > 0 \right\}$$

Fig. 2: Parameter region $\mathfrak{I}$ (shaded) of QMS with Gaussian invariant states.

Remark 4 Condition $\gamma > 0$ indicates an overall higher rate of transitions to lower-level states. In order to interpret the other condition we begin by recalling that the Hamiltonian H has discrete spectrum and the QMS generated by $\mathrm{i}[H, \cdot\,]$ has normal invariant states if and only if $|\kappa|^2 < \Omega^2$. In the case where $\Omega^2 - |\kappa|^2 < 0$ the Hamiltonian H has only continuous spectrum and the additional condition $\gamma^2 > |\kappa|^2 - \Omega^2$ appears. This means that transitions to lower-level states must be stronger to compensate the effect of transitions induced by the Hamiltonian without eigenstates.

Theorem 8 shows that a faithful normal Gaussian invariant state exists and is unique for all parameters $(\gamma, \Omega^2 - |\kappa|^2)$ lying in the open shaded region denoted by $\mathfrak{I}$ (see Fig. 2). We will now show that a normal invariant state, whether Gaussian or not, does *not* exist for any choice of parameters $(\gamma, \Omega^2 - |\kappa|^2)$ lying outside of the region $\mathfrak{I}$.

Equations (4) and (21) suggest that the quantity $\Re(\overline{e^{sZ}z}\, Ce^{sZ}z)$ plays an important role in the existence of invariant states. Therefore we begin by the following two Lemmas, investigating the asymptotic behaviour $e^{t\boldsymbol{Z}}\boldsymbol{z}$ and the convergence of the integral (21).

LEMMA 3 *For all choices of parameters γ, Ω, κ such that $(\gamma, \Omega^2 - |\kappa|^2)$ falls outside the region $\overline{\mathfrak{I}} \setminus \{(0,0)\}$ there exist V_+, a vector subspace of $\mathbb{R}^2$, such that $\left|e^{t\boldsymbol{Z}}\boldsymbol{z}\right|$ diverges as $t \to \infty$ for every $\boldsymbol{z} \in V_+ \setminus \{0\}$.*

Proof. Recall that the matrix $\boldsymbol{Z}$ is given by (18). We can divide the remaining set of parameters in four subsets:

1. $\gamma < 0$ and $\Omega^2 \geq |\kappa|^2$: eigenvalues of $\boldsymbol{Z}$ are $-\gamma \pm \mathrm{i}\sqrt{\Omega^2 - |\kappa|^2}$ both with strictly positive real part,
2. $\gamma \leq 0$ and $\Omega^2 < |\kappa|^2$: eigenvalues of $\boldsymbol{Z}$ are $-\gamma - \sqrt{|\kappa|^2 - \Omega^2} < -\gamma + \sqrt{|\kappa|^2 - \Omega^2}$. At least $-\gamma + \sqrt{|\kappa|^2 - \Omega^2}$ is strictly positive,

3. $\gamma > 0$ and $\gamma^2 + \Omega^2 - |\kappa|^2 < 0$ so that $\Omega^2 - |\kappa|^2 < 0$: eigenvalues of $\boldsymbol{Z}$ are $-\gamma \pm \sqrt{|\kappa|^2 - \Omega^2}$. Only the biggest eigenvalue $-\gamma + \sqrt{|\kappa|^2 - \Omega^2}$ is strictly positive,

4. $\gamma = 0$ and $\Omega = \pm |\kappa|$: the only eigenvalue of $\boldsymbol{Z}$ is 0.

In each of the first three cases there is an eigenvalue λ_+ with positive real part and it is sufficient to choose as V_+ the subspace generated by an eigenvector of λ_+. Indeed if $\boldsymbol{z}_0$ is an eigenvector of λ_+ we have $\left|e^{t\boldsymbol{Z}}\boldsymbol{z}_0\right| = e^{t\Re\lambda_+} |\boldsymbol{z}_0|$.

In the fourth case we have $\boldsymbol{Z} \neq 0$ but $\boldsymbol{Z}^2 = 0$. Hence $e^{t\boldsymbol{Z}} = 1 + t\boldsymbol{Z}$ and there exists $\boldsymbol{z}_0 \in \mathbb{R}^2$ such that $\boldsymbol{Z}\boldsymbol{z}_0 \neq 0$. Therefore

$$\left|e^{t\boldsymbol{Z}}\boldsymbol{z}_0\right| \;=\; |\boldsymbol{z}_0 + t\boldsymbol{Z}\boldsymbol{z}_0| \;\geq\; t\,|\boldsymbol{Z}\boldsymbol{z}_0|$$

and $\left|e^{t\boldsymbol{Z}}\boldsymbol{z}_0\right|$ diverges as $t \to \infty$. It is then sufficient to choose V_+ generated by $\boldsymbol{z}_0$. □

LEMMA 4 *For all choices of parameters γ, Ω, κ such that $(\gamma, \Omega^2 - |\kappa|^2)$ belongs to the boundary of $\mathfrak{I}$ except the origin $(0,0)$ there exists a vector subspace V_+ of $\mathbb{R}^2$ such that for every $\boldsymbol{z} \in V_+ \setminus \{0\}$ the integral*

$$\int_0^t \boldsymbol{z}^T e^{s\boldsymbol{Z}^T} \boldsymbol{C} e^{s\boldsymbol{Z}} \boldsymbol{z} ds \tag{23}$$

diverges as $t \to \infty$.

Proof. Consider first the case $\gamma > 0$, $\gamma = \sqrt{|\kappa|^2 - \Omega^2}$.

Since $\gamma^2 + \Omega^2 - |\kappa|^2 = 0$, $\boldsymbol{Z}$ has 0 as an eigenvalue. Let $\boldsymbol{z}_0$ be an associated eigenvector and fix V_+ as the vector subspace generated by $\boldsymbol{z}_0$. For every $\boldsymbol{z} \in V_+ \setminus \{0\}$ we have $\boldsymbol{z}^T e^{t\boldsymbol{Z}} \boldsymbol{C} e^{t\boldsymbol{Z}} \boldsymbol{z} = \boldsymbol{z}^T \boldsymbol{C} \boldsymbol{z}$. This quantity does not depend on t and is also strictly positive, since $\boldsymbol{C}$ is invertible thanks to Lemma 2. Therefore its integral (23) diverges as $t \to \infty$.

Consider now the case $\gamma = 0$, $\Omega^2 > |\kappa|^2$.

For every such choice of the parameters $\boldsymbol{Z}$ has two distinct eigenvalues, namely $\lambda_\pm = \pm \mathrm{i}\sqrt{\Omega^2 - |\kappa|^2} = \pm \mathrm{i}\delta$ and it can be diagonalized. Let $\boldsymbol{v}_+, \boldsymbol{v}_-$ be two eigenvectors corresponding to $\lambda_\pm$ respectively. If $\boldsymbol{z} = w_- \boldsymbol{v}_+ + w_+ \boldsymbol{v}_-$ we have $e^{t\boldsymbol{Z}}\boldsymbol{z} = w_- e^{\mathrm{i}t\delta}\boldsymbol{v}_+ + w_+ e^{-\mathrm{i}t\delta}\boldsymbol{v}_-$. Now consider the quantity $f_{\boldsymbol{z}}(t) := \boldsymbol{z}^T e^{t\boldsymbol{Z}^T} \boldsymbol{C} e^{t\boldsymbol{Z}} \boldsymbol{z}$ which is non negative and periodic, from the above considerations. We will show $f_{\boldsymbol{z}}(t)$ cannot be identically zero for every $\boldsymbol{z} \in \mathbb{R}^2$. Indeed if it were the case then $f_{\boldsymbol{z}}(0) = 0$ for every $\boldsymbol{z} \in \mathbb{R}^2$ and thus $\boldsymbol{z} \in \ker \boldsymbol{C}$ for every $\boldsymbol{z} \in \mathbb{R}^2$. However $\ker \boldsymbol{C}$ is one-dimensional and there must exists $\boldsymbol{z}_0 \in \mathbb{R}^2$ such that $f_{\boldsymbol{z}_0}(t)$ is not identically zero. Therefore, if V_+ is defined to be the vector subspace generated by $\boldsymbol{z}_0$, the integral (23) diverges, since its argument is a non-negative, periodic function which is not identically zero. □

The previous two Lemmas can now be applied to prove the non-existence of invariant states, for some choices of parameters γ, Ω, κ.

PROPOSITION 4 *If*

$$\text{w*-}\lim_{t\to\infty} \mathcal{T}_t(W(z)) \; = \; 0 \tag{24}$$

(in weak operator topology) for all* $\boldsymbol{z}$ *in a vector subspace of* $\mathbb{R}^2$ *except* $(0,0)$*, then* $\mathcal{T}$ *has no normal invariant state. In particular, for any choice of the parameters* γ, Ω, κ *such that* $(\gamma, \Omega^2 - |\kappa|^2)$ *falls outside of the region* $\mathfrak{I}$ *a normal invariant states for the QMS* $\mathcal{T}$ *does not exist.*

Proof. If ρ is a normal invariant state then for every $z \neq 0$ such that $\boldsymbol{z}$ is in the subspace of $\mathbb{R}^2$ of the hypothesis

$$\mathrm{tr}(\rho W(z)) \; = \; \mathrm{tr}(\rho \mathcal{T}_t(W(z)))$$

for all $t \geq 0$. Taking the limit as $t \to \infty$, by Lemma 3 and Lemma 4 and the explicit formula (4), we get $\mathrm{tr}(\rho W(z)) = 0$. This is a contradiction since $z \to \mathrm{tr}(\rho W(z))$ is continuous and $\mathrm{tr}(\rho W(0)) = 1$.

Observe now that if we are also outside of the region $\overline{\mathfrak{I}} \setminus \{(0,0)\}$ we can use Lemma 3 and fix $\boldsymbol{z}_0 \in V_+ \setminus \{0\}$. Let$f, g \in \mathbb{C}$, thanks to equation (4) we have

$$\left|\langle e(g), W(e^{tZ}z_0)e(f)\rangle\right| \;\; = \;\; \exp\left\{ -\frac{\left|e^{tZ}z_0\right|^2}{2} - \overline{e^{tZ}z_0}f - \bar{g}e^{tZ}z_0 + \bar{g}f \right\}.$$

Since $\left|e^{tZ}z_0\right|$ diverges as $t \to \infty$, we have that $W(e^{tZ}z_0)$ converges weakly to 0. Moreover the Weyl operators are unitary, hence the set $\{W(e^{tZ}z_0) : t \in \mathbb{R}\}$ is bounded and the weak topology coincides with the weak* one. Therefore $W(e^{tZ}z_0)$ converges weakly* to 0 and

$$|\mathrm{tr}(\rho\mathcal{T}_t(W(z_0)))| \; = \; |c_t(z_0)| \left|\mathrm{tr}(\rho W(e^{tZ}z_0))\right| \; \leq \; \left|\mathrm{tr}(\rho W(e^{tZ}z_0))\right| ,$$

where $c_t(z_0)$ is the constant multiplying the Weyl operator in (4). This means $\mathcal{T}_t(W(z_0))$ converges to 0 in the weak* topology for every $z_0 \in V_+ \setminus \{0\}$. Hence there can be no normal invariant states.

Suppose we are now in the region $\partial\mathfrak{I} \setminus \{0\}$. Let $\boldsymbol{z}_0 \in V_+$, whose existence is given by Lemma 4. Thanks to (4) one has

$$|\mathrm{tr}(\rho\mathcal{T}_t(W(z_0)))| \; \leq \; \exp\left\{ -\frac{1}{2}\int_0^t \boldsymbol{z_0}^T e^{s\boldsymbol{Z}^T} \boldsymbol{C} e^{s\boldsymbol{Z}} \boldsymbol{z}_0 ds \right\},$$

since $|\mathrm{tr}(\rho W(z))| \leq 1$ for every $z \in \mathbb{C}$. Letting $t \to \infty$ one has $\mathcal{T}_t(W(z_0)) \to 0$ in the weak* topology for every $z_0 \in V_+ \setminus \{0\}$. Hence also in this case there are no normal invariant states. □

Summarizing we proved the following complement to Theorem 8.

THEOREM 9 *Let $(\mathcal{T}_t)_{t\geq 0}$ be the QMS with GKSL generator associated with H, L_1, L_2 as in (2), (3) or with H and a single Kraus operator. The QMS $\mathcal{T}$ has a normal invariant state if and only if $\gamma > 0$ and $\gamma^2 + \Omega^2 - |\kappa|^2 > 0$. The normal invariant state is also unique.*

Moreover, we also showed that the above invariant states are either faithful or pure.

PROPOSITION 5 *The invariant state given by Theorem 8 is pure if and only if $\Omega^2 - |\kappa|^2 > 0$, there is a single Kraus operator $L = \bar{v}a + ua^\dagger$ and*

$$\frac{\Im(uv)}{|u-\bar{v}|^2} = \frac{\Im(\kappa)}{2(\Omega - \Re(\kappa))}, \qquad \frac{\gamma}{\sqrt{\Omega^2 - |\kappa|^2}} = \frac{|u-\bar{v}|^2}{2\,|\Re(\kappa) - \Omega|}. \tag{25}$$

In all the other cases it is faithful.

Proof. A Gaussian state is faithful if and only if $\boldsymbol{S} - \mathrm{i}\boldsymbol{J} > 0$ otherwise it is pure (see [24, Sect. 2] and also [25]). As in the proof of Theorem 8 we can use

$$\boldsymbol{S} - \mathrm{i}\boldsymbol{J} = \int_0^\infty e^{s\boldsymbol{Z}^T} \left(\boldsymbol{C} + \mathrm{i}\left(\boldsymbol{Z}^T\boldsymbol{J} + \boldsymbol{J}\boldsymbol{Z}\right)\right) e^{s\boldsymbol{Z}} \mathrm{d}s\,, \tag{26}$$

and study its kernel. Clearly, if $\boldsymbol{C} + \mathrm{i}\left(\boldsymbol{Z}^T\boldsymbol{J} + \boldsymbol{J}\boldsymbol{Z}\right) > 0$, also $\boldsymbol{S} - \mathrm{i}\boldsymbol{J} > 0$, since $e^{s\boldsymbol{Z}}$ is invertible. This happens whenever there are two Kraus operators, thanks to Lemma 2. So the state can be pure only if there is a single Kraus operator. We restrict ourselves to this case. Now the kernel of $\boldsymbol{S} - \mathrm{i}\boldsymbol{J}$ is non-trivial if and only if the argument of the integral (26). has a nontrivial kernel. This happens whenever at least one of the eigenvectors of $\boldsymbol{Z}$ belongs to $\ker(\boldsymbol{C}+\mathrm{i}(\boldsymbol{Z}^T\boldsymbol{J}+\boldsymbol{J}\boldsymbol{Z}))$. Indeed suppose there is $\boldsymbol{z}_0 \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that $(\boldsymbol{C}+\mathrm{i}(\boldsymbol{Z}^T\boldsymbol{J}+\boldsymbol{J}\boldsymbol{Z}))e^{t\boldsymbol{Z}}\boldsymbol{z}_0 = 0$ for all $t \geq 0$. Since $\ker(\boldsymbol{C}+\mathrm{i}(\boldsymbol{Z}^T\boldsymbol{J}+\boldsymbol{J}\boldsymbol{Z}))$ is one-dimensional suppose it is generated by $\boldsymbol{v}_0 \in \mathbb{R}^2$, we have then $e^{t\boldsymbol{Z}}\boldsymbol{z}_0 = \lambda_t \boldsymbol{v}_0$ for some $\lambda_t \in \mathbb{R}$. In particular $\lambda_{s+t}\boldsymbol{v}_0 = e^{(t+s)\boldsymbol{Z}}\boldsymbol{z}_0 = \lambda_s e^{t\boldsymbol{Z}}\boldsymbol{v}_0$ which means $\boldsymbol{v}_0$ is an eigenvector for $e^{t\boldsymbol{Z}}$ and therefore it is also an eigenvector for $\boldsymbol{Z}$. The converse implication is trivial.

Suppose $\Omega \neq \Re(\kappa)$, via explicit calculations the eigenvectors of $\boldsymbol{Z}$ are $\boldsymbol{v}_\pm = (\Re(\kappa) - \Omega, \Im(\kappa) \pm \sqrt{|\kappa|^2 - \Omega^2})$. We have

$$\begin{aligned}&(\boldsymbol{C} + \mathrm{i}(\boldsymbol{Z}^T\boldsymbol{J} + \boldsymbol{J}\boldsymbol{Z}))\boldsymbol{v}_\pm \\ &= \begin{bmatrix} |u+\bar{v}|^2\,(\Re(\kappa) - \Omega) + 2\left(\Im(\kappa) \pm \sqrt{|\kappa|^2 - \Omega^2}\right)(\Im(uv) - \mathrm{i}\gamma) \\ \left(\Im(\kappa) \pm \sqrt{|\kappa|^2 - \Omega^2}\right)|u-\bar{v}|^2 + 2\,(\Re(\kappa) - \Omega)\,(\Im(uv) + \mathrm{i}\gamma) \end{bmatrix}.\end{aligned} \tag{27}$$

Now if this were the null vector the imaginary part of the second entry should be zero. If $|\kappa|^2 - \Omega^2 \geq 0$ this would imply $\gamma(\Re(\kappa) - \Omega) = 0$ which is impossible since $\gamma > 0$ and we supposed $\Re(\kappa) \neq \Omega$. Therefore, for (27) to be zero, $|\kappa|^2 - \Omega^2 < 0$ and, letting $\delta := \sqrt{\Omega^2 - |\kappa|^2}$, this is equivalent to

$$\begin{cases} |u+\bar{v}|^2 \left(\Re(\kappa) - \Omega\right) + 2\Im(\kappa)\Im(uv) \pm 2\delta\gamma \;=\; \pm\delta\Im(uv) - \gamma\Im(\kappa) \;=\; 0 \\ \Im(\kappa)\left|u-\bar{v}\right|^2 + 2(\Re(\kappa)-\Omega)\Im(uv) \;=\; \pm\delta\left|u-\bar{v}\right|^2 + 2\gamma(\Re(\kappa)-\Omega) \;=\; 0\,. \end{cases}$$

Those equation are in turn equivalent to

$$\frac{\Im(uv)}{|u-\bar{v}|^2} \;=\; -\frac{\Im(\kappa)}{2(\Re(\kappa)-\Omega)}\,, \qquad \frac{\gamma}{\delta} \;=\; \mp\frac{|u-\bar{v}|^2}{2(\Re(\kappa)-\Omega)}\,,$$

that lead to , having chosen $\boldsymbol{v}_\pm$ in order to have $\gamma > 0$. Vice versa if u, v, κ, Ω satisfy equations then one of $\boldsymbol{v}_\pm$ is in $\ker(\boldsymbol{C} + \mathrm{i}(\boldsymbol{Z}^T\boldsymbol{J} + \boldsymbol{J}\boldsymbol{Z}))$.

Suppose now $\Omega = \Re(\kappa)$. The proper eigenvectors of $\boldsymbol{Z}$ are $\boldsymbol{v}_1 = (0,1), \boldsymbol{v}_2 = (\Im(\kappa), \Re(\kappa))$ if $\Im(\kappa) \neq 0$ or $\boldsymbol{v}_1 = (0,1), \boldsymbol{v}_3 = (1,0)$ if $\Im(\kappa) = 0$. One has

$$\begin{aligned} \Im((\boldsymbol{C} + \mathrm{i}(\boldsymbol{Z}^T\boldsymbol{J} + \boldsymbol{J}\boldsymbol{Z}))\boldsymbol{v}_1) &= \begin{bmatrix} -2\gamma \\ 0 \end{bmatrix}, \\ \Im((\boldsymbol{C} + \mathrm{i}(\boldsymbol{Z}^T\boldsymbol{J} + \boldsymbol{J}\boldsymbol{Z}))\boldsymbol{v}_2) &= \begin{bmatrix} -2\gamma\Re(\kappa) \\ 2\gamma\Im(\kappa) \end{bmatrix}, \\ \Im((\boldsymbol{C} + \mathrm{i}(\boldsymbol{Z}^T\boldsymbol{J} + \boldsymbol{J}\boldsymbol{Z}))\boldsymbol{v}_3) &= \begin{bmatrix} 0 \\ 2\gamma \end{bmatrix}, \end{aligned}$$

that cannot both be zero since this would require either $\gamma = 0$ or $\kappa = 0$, which would imply v_2 is the null vector. □

7. Examples

In this section we present the application of our results in two remarkable cases. These also serve to illustrate the relationships we have found between the parameters that determine the behaviour of the dynamics.

7.1. Open quantum harmonic oscillator

Let $\mathcal{T}$ be the QMS with generator in a generalized GKSL form with

$$L_1 = \mu\, a\,, \quad L_2 = \lambda\, a^\dagger, \quad H \;=\; \Omega\, a^\dagger a + \frac{\kappa}{2}a^{\dagger 2} + \frac{\overline{\kappa}}{2}a^2 + \frac{\zeta}{2}a^\dagger + \frac{\overline{\zeta}}{2}a \qquad (28)$$

with $\lambda, \mu \geq 0$, $\Omega \in \mathbb{R}$, $\kappa, \zeta \in \mathbb{C}$. The special case where $\kappa = \zeta = 0$ has been analyzed in [7] providing the full spectral analysis of the generator $\mathcal{L}$ in the L^2 space of the invariant state for $\lambda < \mu$.

In this model $\gamma = (\mu^2 - \lambda^2)/2$. Moreover, in the case where both λ, μ are strictly positive, the QMS is irreducible (Theorem 5) and admits a unique faithful normal invariant state if and only if $\lambda^2 < \mu^2$ and $\left(\mu^2 - \lambda^2\right)/4 + \Omega^2 - |\kappa|^2 > 0$ (Theorem 8) with the explicit mean vector ω and covariance operator S as in (21).

If $\mu = 0$ and $\lambda > 0$ we obtain a QMS which is irreducible if and only if $\overline{\kappa}\lambda^2 \neq 0$, namely $\kappa \neq 0$ and has no normal invariant state (Sect. 4.2).

Finally, in the case where $\lambda = 0$ and $\mu > 0$ we find a QMS which is irreducible if and only if $\kappa\mu^2 \neq 0$, i.e., $\kappa \neq 0$. It admits invariant states if and only if $|\kappa|^2 < \Omega^2 + \lambda^4/4$; these will be faithful if $\Re(\kappa) \neq 0$ and pure otherwise (Sect. 4.1). For any initial state ρ_0, in both cases, $t^{-1}\int_0^t \mathcal{T}_{*s}(\rho_0)\mathrm{d}s$ converges towards the unique invariant state by Theorem 8.

It is worth noticing here that the Hamiltonian H is bounded from below or above if and only if $\Omega^2 - |\kappa|^2 \geq 0$, in which case it has discrete spectrum. Therefore condition $|\kappa|^2 < \Omega^2 + \lambda^4/4$ appears as a relaxation of discreteness of spectrum that allows existence of normal invariant states.

7.2. Quantum Fokker-Planck model

The quantum Fokker–Planck (QFP) model is an open quantum system introduced to describe the quantum mechanical charge-transport including diffusive effects (see [2, 21, 26] and the references therein). In this subsection we show that a simple application of our results allows one to study the dynamics.

The formal generator

$$\begin{aligned} \mathcal{L}(x) &= \frac{\mathrm{i}}{2}\left[p^2 + \omega^2 q^2, x\right] + \mathrm{i}g\left\{p, [q, x]\right\} \\ &\quad - D_{qq}[p, [p, x]] - D_{pp}[q, [q, x]] + 2D_{pq}[q, [p, x]]\,, \end{aligned}$$

can be written in generalized GKSL form (1) with

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2 + g(pq + qp)\right),$$

and L_1, L_2 are the operators

$$L_1 = \frac{-2D_{pq} + \mathrm{i}g}{\sqrt{2D_{pp}}}p + \sqrt{2D_{pp}}\,q\,, \qquad L_2 = \frac{2\sqrt{\Delta}}{\sqrt{2D_{pp}}}\,p\,,$$

where $\omega^2 > 0$, $D_{pp} > 0, D_{qq} \geq 0$, $D_{pq} \in \mathbb{R}$ and $\Delta = D_{pp}D_{qq} - D_{pq}^2 - g^2/4 \geq 0$. Clearly, L_1, L_2 are linearly independent if and only if $\Delta > 0$. Moreover,

$$\begin{aligned} L_1 &= \frac{-2\mathrm{i}D_{pq} - g}{2\sqrt{D_{pp}}}(a^\dagger - a) + \sqrt{D_{pp}}(a^\dagger + a)\,, \\ L_2 &= \frac{\mathrm{i}\sqrt{\Delta}}{\sqrt{D_{pp}}}(a^\dagger - a)\,, \\ H &= \frac{\omega^2+1}{2}aa^\dagger + \frac{\omega^2 - 1 + 2\mathrm{i}g}{4}\,a^{\dagger 2} + \frac{\omega^2 - 1 - 2\mathrm{i}g}{4}\,a^2 + \frac{\omega^2+1}{4} \end{aligned}$$

so that

$$\overline{v}_1 = \frac{2\mathrm{i}D_{pq} + g}{2\sqrt{D_{pp}}} + \sqrt{D_{pp}}\,, \quad u_1 = -\frac{2\mathrm{i}D_{pq} + g}{2\sqrt{D_{pp}}} + \sqrt{D_{pp}}\,, \quad \overline{v}_2 = -\frac{\mathrm{i}\sqrt{\Delta}}{\sqrt{D_{pp}}}\,,$$
$$u_2 = \frac{\mathrm{i}\sqrt{\Delta}}{\sqrt{D_{pp}}}\,, \quad \Omega = \frac{\omega^2+1}{2}\,, \quad \kappa = \frac{\omega^2-1}{2} + \mathrm{i}g\,.$$

Compute

$$\gamma \;=\; \frac{1}{2}\sum_{\ell=1}^{2}\left(|v_\ell|^2 - |u_\ell|^2\right) \;=\; g\,, \quad \Omega^2 - |\kappa|^2 \;=\; \omega^2 - g^2\,, \quad \gamma^2 + \Omega^2 - |\kappa|^2 \;=\; \omega^2\,.$$

Therefore, in the case where $\Delta > 0$ Kraus operators L_1, L_2 are linearly independent, the QFP semigroup is irreducible and a Gaussian invariant state exists if and only if $g = \gamma > 0$. This is given explicitly in Theorem 8. Moreover, it is also faithful and it is the unique normal invariant state by irreducibility.

The case $\Delta = 0$ has to be considered separately (see [2, 26]). By Theorem (7) the QFP semigroup is irreducible if and only if $2\Omega\overline{v}_1 u_1 = \kappa\overline{v}_1^2 + \overline{\kappa}u_1^2$. Taking the imaginary parts of this identity we find

$$gD_{pp} \;=\; -\omega^2 D_{pq}\,. \tag{29}$$

Taking real parts we find $\omega^2\left(4D_{pq}^2 - g^2\right) + 4D_{pp}^2 = -8gD_{pq}D_{pp}$ and, from (29), we find the identity

$$\omega^2\left(4D_{pq}^2 - g^2\right) + 4D_{pp}^2 \;=\; -8gD_{pq}D_{pp} \;=\; 8\omega^2 D_{pq}^2$$

namely $4D_{pp}^2 = \omega^2\left(4D_{pq}^2 + g^2\right)$ and, by $\Delta = 0$ together with $D_{pp} > 0$

$$D_{pp} \;=\; \omega^2 D_{qq}\,. \tag{30}$$

Note that $\Delta = 0$ together with (29) and (30) are equivalent to conditions under which, for $\gamma > 0$, the Gaussian normal invariant state of the QFP model is pure (see [2, Lemma 9.1]).

Clearly, they are equivalent to (25). Indeed, a straightforward computation shows that the first identity is equivalent to $D_{pq} = -gD_{qq}$ and the second one to $D_{qq} = g/(2\delta)$ which follows from $\Delta = 0$, (29) and (30) (see [2, Lemma 9.1] for details).

Convergence towards the unique invariant state of $t^{-1} \int_0^t \mathcal{T}_{*t}(\rho_0)\mathrm{d}s$ holds for any initial state ρ_0 by Theorem 8.

8. Conclusions and Outlook

We considered the most general Gaussian QMS on the one mode Fock space $\Gamma(\mathbb{C})$ of the regular representation of one-dimensional CCR. The GKSL generator associated with unbounded operators (2) and (3) depends on 7 parameters (or 5 in the case where there is only one noise operator). We presented its construction starting form the unbounded generator and proved the known explicit formula for the action Weyl operators. We characterized irreducibility in terms of parameters of the model. This property always holds true when there are two linearly independent noise operators L_1, L_2. However, if there is only a single noise operator L_1, irreducibility holds if and only if the operators L_1 and $[H, L_1]$ are linearly independent (the Hörmander type commutator condition that appears in many fields of mathematics, from partial differential equations to control theory). Finally, still in terms of the parameters of the model, we established the necessary and sufficient condition $\gamma > 0$ and $\gamma^2 + \Omega^2 - |\kappa|^2 > 0$ for existence and uniqueness of normal invariant states. This condition also implies, by irreducibility convergence towards the unique invariant state.

It would be useful and interesting to extend the above results to Gaussian QMS on the algebra of bounded operators on d-mode Fock spaces. The explicit formula for the action on Weyl operators is known also in this case. We guess that the equivalence of irreducibility with an Hörmander type commutator condition can be proved as well considering commutators of H and noise operators L_ℓ of order up to $2d-1$. This advance seems to require a deep study of regularity properties of Gaussian semigroups as in the classical case. However, we think that it is a necessary step in order to establish precise relationships between the behaviour of the infinite dimensional QMS and the $2d \times 2d$ dimensional matrices Z and C and reduce a lot of infinite dimensional problems on the dynamics to finite dimensional ones on matrices. Results will be the object of a forthcoming paper.

Appendix A

In this section we prove Theorem 4. We begin with the following lemma.

LEMMA 5 *For all* $\xi \in \mathrm{Dom}(N^2)$ *and all* $\theta \in \mathbb{R}$ *we have*

$$\begin{aligned}\left\|(e^{\mathrm{i}\theta}a^\dagger + e^{-\mathrm{i}\theta}a)\xi\right\|^2 &\le 2\left\|(aa^\dagger + a^\dagger a)^{1/2}\xi\right\|^2 \\ \left\|(e^{\mathrm{i}\theta}a^{\dagger 2} + e^{-\mathrm{i}\theta}a^2)\xi\right\|^2 &\le \left\|(aa^\dagger + a^\dagger a)\xi\right\|^2 + 3\left\|\xi\right\|^2 .\end{aligned}$$

Proof. Computations below should be done on quadratic forms defined on the domain $D \times D$. However, we do only the algebraic computations to simplify the notation.

To prove the first inequality we begin by expanding

$$0 \le |e^{\mathrm{i}\theta}a^\dagger - e^{-\mathrm{i}\theta}a|^2 = a^\dagger a - e^{2\mathrm{i}\theta}a^{\dagger 2} - e^{-2\mathrm{i}\theta}a^2 + aa^\dagger$$

which implies

$$e^{2\mathrm{i}\theta}a^{\dagger 2} + e^{-2\mathrm{i}\theta}a^2 \le a^\dagger a + aa^\dagger .$$

It follows that

$$|e^{\mathrm{i}\theta}a^\dagger + e^{-\mathrm{i}\theta}a|^2 \le 2(a^\dagger a + aa^\dagger)$$

and the first inequality is proved. To prove the second inequality, first note that

$$0 \le |e^{\mathrm{i}\theta}a^{\dagger 2} - e^{-\mathrm{i}\theta}a^2|^2 = a^2a^{\dagger 2} - e^{2\mathrm{i}\theta}a^{\dagger 4} - e^{-2\mathrm{i}\theta}a^4 + a^{\dagger 2}a^2$$

and so

$$e^{2\mathrm{i}\theta}a^{\dagger 4} + e^{-2\mathrm{i}\theta}a^4 \le a^2a^{\dagger 2} + a^{\dagger 2}a^2 .$$

Now

$$\begin{aligned}&(e^{\mathrm{i}\theta}a^{\dagger 2} + e^{-\mathrm{i}\theta}a^2)^2 - (aa^\dagger + a^\dagger a)^2 = e^{2\mathrm{i}\theta}a^{\dagger 4} + a^{\dagger 2}a^2 + a^2a^{\dagger 2} + e^{-2\mathrm{i}\theta}a^4 \\ &- (aa^\dagger)^2 - (a^\dagger a)^2 - aa^{\dagger 2}a - a^\dagger a^2 a^\dagger \\ &\le 2a^{\dagger 2}a^2 + 2a^2a^{\dagger 2} - (aa^\dagger)^2 - (a^\dagger a)^2 - aa^{\dagger 2}a - a^\dagger a^2 a^\dagger .\end{aligned}$$

The right-hand side is equal to

$$2N(N-1) + 2(N+1)(N+2) - (N+1)^2 - N^2 - (N+1)N - N(N+1) = 3$$

and so

$$(e^{\mathrm{i}\theta}a^{\dagger 2} + e^{-\mathrm{i}\theta}a^2)^2 \le (aa^\dagger + a^\dagger a)^2 + 3 .$$

The claimed inequality readily follows. □

We will show that the graph norms of G, G_0 and N are equivalent. To this end need two preliminary lemmas.

LEMMA 6 *Let λ_0 be the smallest eigenvalue of the 2×2 matrix*

$$\begin{bmatrix} v_1 & v_2 \\ \overline{u}_1 & \overline{u}_2 \end{bmatrix} \cdot \begin{bmatrix} \overline{v}_1 & u_1 \\ \overline{v}_2 & u_2 \end{bmatrix}$$

which is strictly positive by the linear independence of L_1, L_2. There exists a constant $c_1 > 0$ depending on v_1, u_1, v_2, u_2 and uniformly bounded for v_1, u_1, v_2, u_2 in a bounded subset of $\mathbb{C}^4$ such that

$$(-2G_0)^2 \;\geq\; \lambda_0^2 (a^\dagger a + a\, a^\dagger)^2 - c_1 (a^\dagger a + a\, a^\dagger)\,.$$

Proof. Since $-2G_0 = L_1^* L_1 + L_2^* L_2$,

$$G_0 \;=\; -\frac{1}{2}\sum_{\ell=1}^{2}\left(\left(|v_\ell|^2 a^\dagger a + |u_\ell|^2 a a^\dagger\right) + v_\ell u_\ell a^{\dagger 2} + \overline{v}_\ell \overline{u}_\ell a^2\right)$$

for all $\xi \in D$, thinking of $(a\,\xi, a^\dagger \xi)$ as a vector in $\mathsf{h}\oplus\mathsf{h}$ and of product of a row vector with a column vector as the natural scalar product in $\mathsf{h}\oplus\mathsf{h}$, we can write $\langle \xi, G_0 \xi\rangle$ as follows

$$\langle \xi, G_0\xi\rangle \;=\; -\frac{1}{2}[a^\dagger\xi, a\,\xi] \begin{bmatrix} v_1 & v_2 \\ \overline{u}_1 & \overline{u}_2 \end{bmatrix} \begin{bmatrix} \overline{v}_1 & u_1 \\ \overline{v}_2 & u_2 \end{bmatrix} \begin{bmatrix} a\,\xi \\ a^\dagger \xi \end{bmatrix}.$$

This notation is typical in the study of quadratic Hamiltonians (see, for instance, [9, 27, 28, 29]). Recall that, by linear independence of L_1, L_2, the above matrices have non-zero determinant. Therefore their product is *strictly* positive definite and, calling λ_1 its biggest eigenvalue, we have

$$\lambda_1 \left\langle \xi, (a^\dagger a + a\, a^\dagger)\xi \right\rangle \;\geq\; \langle \xi, -2G_0\xi\rangle \;\geq\; \lambda_0 \left\langle \xi, (a^\dagger a + a\, a^\dagger)\xi \right\rangle\,. \tag{31}$$

In a similar way, dropping the vector ξ and denoting by l.o.t. monomials of order 2 or less in creation and annihilation operators we have the inequalities

$$\begin{aligned}
(-2G_0)^2 \;&=\; \sum_\ell L_\ell^* \,(-2G_0)\, L_\ell + \text{l.o.t.} \\
&=\; [a^\dagger, a] \begin{bmatrix} v_1 & v_2 \\ \overline{u}_1 & \overline{u}_2 \end{bmatrix} \begin{bmatrix} -2G_0 & 0 \\ 0 & -2G_0 \end{bmatrix} \begin{bmatrix} \overline{v}_1 & u_1 \\ \overline{v}_2 & u_2 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} + \text{l.o.t.} \\
&\geq\; \lambda_0\, [a^\dagger, a] \begin{bmatrix} v_1 & v_2 \\ \overline{u}_1 & \overline{u}_2 \end{bmatrix} \begin{bmatrix} a^\dagger a + a a^\dagger & 0 \\ 0 & a^\dagger a + a a^\dagger \end{bmatrix} \begin{bmatrix} \overline{v}_1 & u_1 \\ \overline{v}_2 & u_2 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} + \text{l.o.t.} \\
&=\; \lambda_0\, a\, [a^\dagger, a] \begin{bmatrix} v_1 & v_2 \\ \overline{u}_1 & \overline{u}_2 \end{bmatrix} \begin{bmatrix} \overline{v}_1 & u_1 \\ \overline{v}_2 & u_2 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} a^\dagger \\
&\qquad + \lambda_0\, a^\dagger\, [a^\dagger, a] \begin{bmatrix} v_1 & v_2 \\ \overline{u}_1 & \overline{u}_2 \end{bmatrix} \begin{bmatrix} \overline{v}_1 & u_1 \\ \overline{v}_2 & u_2 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} a + \text{l.o.t.} \\
&=\; \lambda_0\, a(-2G_0)a^\dagger + \lambda_0\, a^\dagger(-2G_0)a + \text{l.o.t.} \\
&\geq\; \lambda_0^2\, a(a^\dagger a + a a^\dagger)a^\dagger + \lambda_0^2\, a^\dagger(a^\dagger a + a a^\dagger)a + \text{l.o.t.} \\
&=\; \lambda_0^2 (a^\dagger a + a a^\dagger)^2 + \text{l.o.t.}
\end{aligned}$$

Lower order terms can be controlled in terms of $(2N+1) = (a^\dagger a + a\,a^\dagger)$ by Lemma 5 and the proof is complete. □

LEMMA 7 *The commutator* $[H, G_0]$ *is a second order degree polynomial in* $a, a^\dagger$ *and*

$$|\langle \xi, [H, G_0]\xi\rangle| \;\leq\; c_2\langle \xi, (a^\dagger a + a\,a^\dagger)^{1/2}\xi\rangle$$

for some constant $c_2 > 0$ *depending on all parameters in the model.*

Proof. A long but straightforward computation yields (summation on $\ell = 1, 2$ is implicit)

$$\begin{aligned}
[H, G_0] \;&=\; \mathrm{i}\Im(\kappa\,(\overline{v}_\ell \overline{u}_\ell))(a^\dagger a + aa^\dagger) + \Big(\Omega\,(\overline{v}_\ell \overline{u}_\ell) - \frac{\overline{\kappa}}{2}\,(|v_\ell|^2 + |u_\ell|^2)\,\Big)a^2 \\
&+\; \Big(- \Omega\,(v_\ell u_\ell) + \frac{\kappa}{2}\,(|v_\ell|^2 + |u_\ell|^2)\,\Big)a^{\dagger 2} \\
&+ \Big(\frac{\zeta}{2}\,(\overline{v}_\ell \overline{u}_\ell) - \frac{\overline{\zeta}}{2}\,(|v_\ell|^2 + |u_\ell|^2)\,\Big)a + \Big(- \frac{\overline{\zeta}}{2}\,(v_\ell u_\ell) + \frac{\zeta}{2}\,(|v_\ell|^2 + |u_\ell|^2)\,\Big)a^\dagger .
\end{aligned}$$

The claimed inequality follows from Lemma 5 and the Schwarz inequality. □

Proof of Theorem 4. Clearly $\mathrm{Dom}(N)$ is contained in $\mathrm{Dom}(G_0)$ and $\mathrm{Dom}(G)$.

In order to prove the opposite inclusion we show that there exist constants c_3, c_4 such that $\|N\xi\|^2 \leq c_3\|G_0\xi\|^2 + c_4\|\xi\|^2$ for all $\xi \in D$. The conclusion follows because D is an essential domain for G_0 and G by their definition.

For all $\xi \in D$, $\epsilon > 0$ by Lemma 6 and the Young's inequality, we have the following inequalities

$$\begin{aligned}
\|G_0\xi\|^2 \;&=\; \langle \xi, G_0^2\xi\rangle \\
&\geq\; \frac{\lambda_0^2}{4}\left\langle \xi, (a^\dagger a + a\,a^\dagger)^2\xi\right\rangle - \frac{c_1}{4}\left\langle \xi, (a^\dagger a + a\,a^\dagger)\xi\right\rangle \\
&\geq\; \frac{\lambda_0^2}{4}\left\langle \xi, (a^\dagger a + a\,a^\dagger)^2\xi\right\rangle - \frac{c_1}{4}\,\|\xi\|\cdot\left\|\left(a^\dagger a + a\,a^\dagger\right)\xi\right\| \\
&\geq\; \frac{\lambda_0^2}{4}\left\langle \xi, (a^\dagger a + a\,a^\dagger)^2\xi\right\rangle - \frac{\lambda_0^2}{8}\left\|(a^\dagger a + a\,a^\dagger)\xi\right\|^2 - \frac{c_1^2}{8\lambda_0^2}\,\|\xi\|^2 \\
&=\; \frac{\lambda_0^2}{8}\left\|(a^\dagger a + a\,a^\dagger)\xi\right\|^2 - \frac{c_1^2}{8\lambda_0^2}\,\|\xi\|^2 .
\end{aligned}$$

Since $\left\|\left(a^\dagger a + a\,a^\dagger\right)\xi\right\|^2 \geq 4\,\|N\xi\|^2$ we find the inequality

$$\|N\xi\|^2 \;\leq\; \frac{2}{\lambda_0^2}\|G_0\xi\|^2 + \frac{c_1^2}{4\lambda_0^4}\,\|\xi\|^2 \tag{32}$$

for all $\xi \in D$ implying that $\mathrm{Dom}(G_0) \subseteq \mathrm{Dom}(N)$.

In order to prove that the domain of G is also contained in the domain of N note that $G = G_0 - \mathrm{i}H$ on D and write

$$\|G\xi\|^2 \;=\; \langle \xi, (G_0 + \mathrm{i}H)(G_0 - \mathrm{i}H)\xi\rangle \;=\; \left\langle \xi, (G_0^2 + H^2)\xi\right\rangle + \mathrm{i}\,\langle \xi, [H, G_0]\xi\rangle\,. \tag{33}$$

Now by Lemma 7, $\left\langle \xi, H^2\xi\right\rangle \geq 0$ and the previous inequality (32) we find

$$\begin{aligned} \|G\xi\|^2 &\geq \langle \xi, G_0^2\xi\rangle - c_2\left\langle \xi, (a^\dagger a + a\,a^\dagger)^{1/2}\xi\right\rangle \\ &\geq \frac{\lambda_0^2}{2}\,\|N\xi\|^2 - c_2\sqrt{2}\,\|\xi\|\cdot\|N^{1/2}\xi\| - \frac{c_1^2}{4}\,\|\xi\|^2\,. \end{aligned}$$

We can now proceed as in the final part of the proof of (32) with an application of the Young inequality to show that $\mathrm{Dom}(G) \subseteq \mathrm{Dom}(N)$. □

Appendix B

In this appendix we prove Lemma 1. We begin by noting that $\mathrm{Dom}(N) \subseteq \mathrm{Dom}(G_0)$.

Conversely, note that for all $r \in \mathbb{R}$, on the domain $\mathrm{Dom}(N)$ of the number operator, in a natural matrix notation, we have

$$\begin{aligned} L^*L &= |v|^2 a^\dagger a + vu\,a^{\dagger 2} + \overline{vu}\,a^2 + |u|^2 aa^\dagger \\ &= \left(|v|^2 + r\right) a^\dagger a + vua^{\dagger 2} + \overline{vu}a^2 + \left(|u|^2 - r\right) aa^\dagger + r\mathbb{1} \\ &= \begin{bmatrix} a^\dagger & a \end{bmatrix} \begin{bmatrix} |v|^2 + r & vu \\ \overline{vu} & |u|^2 - r \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} + r\mathbb{1} \end{aligned}$$

The trace of the above 2×2 matrix is strictly positive and the determinant

$$r\left(|u|^2 - |v|^2\right) - r^2$$

if we choose $r = \left(|u|^2 - |v|^2\right)/2$, it is equal to $\left(|u|^2 - |v|^2\right)^2/4 > 0$ and the lowest eigenvalue is $\left(|v| - |u|\right)^2/2$. It follows that

$$L^*L \geq \frac{(|v| - |u|)^2}{2}\left(aa^\dagger + a^\dagger a\right) + \frac{|v|^2 - |u|^2}{2}\mathbb{1}$$

and, denoting by l.o.t. monomials of order 2 or less in creation and annihilation operators,

$$\begin{aligned} (L^*L)^2 &= L^*(LL^*)L = L^*(L^*L)L + \text{l.o.t.} \\ &\geq \frac{1}{2}\left(|v| - |u|\right)^2 L^*\left(aa^\dagger + a^\dagger a\right) L + \text{l.o.t.} \\ &= \frac{1}{2}\left(|v| - |u|\right)^2\left(aL^*La^\dagger + a^\dagger L^*La\right) + \text{l.o.t.} \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{4}(|v|-|u|)^4 \left(a\left(aa^\dagger + a^\dagger a\right)a^\dagger + a^\dagger\left(aa^\dagger + a^\dagger a\right)a\right) + \text{l.o.t.} \\ &= (|v|-|u|)^4 \left(a^\dagger a\right)^2 + \text{l.o.t.} \end{aligned}$$

Therefore there exists a constant $c > 0$ such that

$$(|v|-|u|)^4 \, \|a^\dagger a\,\xi\|^2 \leq \|L^*L\,\xi\|^2 + c\,\|\xi\|^2 \tag{34}$$

for all $\xi \in D$ and $\mathrm{Dom}(L^*L) \subseteq \mathrm{Dom}(N)$. This shows the identity $\mathrm{Dom}(G_0) = \mathrm{Dom}(N)$.

In order to prove the other one, note first that $\mathrm{Dom}(N) \subseteq \mathrm{Dom}(G)$. Then, for all $\xi \in D$, compute

$$\|G\xi\|^2 = \|G_0\xi\|^2 + \|H\xi\|^2 + \langle \xi, \mathrm{i}[H, G_0]\xi\rangle \,.$$

Since the commutator $[H, G_0]$ is a second order polynomial in $a, a^\dagger$ there exists a constant $c' > 0$ such that $\langle \xi, \mathrm{i}[H, G_0]\xi\rangle \geq -c'\|N^{1/2}\xi\|^2$. Recalling (34), by the Young inequality, we have

$$\begin{aligned} \|G\xi\|^2 &\geq \|G_0\xi\|^2 - c'\|N^{1/2}\xi\|^2 \\ &\geq \frac{(|v|-|u|)^4}{4}\|a^\dagger a\xi\|^2 - c\|\xi\|^2 - \frac{(|v|-|u|)^4}{8}\|a^\dagger a\xi\|^2 - \frac{4}{c'^2\,(|v|-|u|)^4}\|\xi\|^2 \\ &= \frac{(|v|-|u|)^4}{8}\|a^\dagger a\xi\|^2 - c''\|\xi\|^2 \end{aligned}$$

where c'' is another constant. Thus $\mathrm{Dom}(G) \subseteq \mathrm{Dom}(N)$ and the proof of Lemma 1 is complete.

Appendix C

LEMMA 8 *Let $\mu, \lambda, x, y \in \mathbb{R}$ with $\lambda \neq 0$. For all $r > 0$ and $w > 0$ such that $w < \min\{1, (2x^2)^{-1}\}$ there exists $\epsilon > 0$ such that*

$$\begin{bmatrix} \mu^2 + r^4 & \mu x & \lambda\mu \\ \mu x & x^2 & \lambda x \\ \lambda\mu & \lambda x & \lambda^2 \end{bmatrix} \geq \epsilon \begin{bmatrix} r^4 & 0 & 1 \\ 0 & -1/2 & 0 \\ 1 & 0 & \lambda^2 w \end{bmatrix}$$

Proof. The difference of the above matrices is

$$\begin{bmatrix} \mu^2 + r^4(1-\epsilon) & \mu x & \lambda\mu - \epsilon \\ \mu x & x^2 + \epsilon/2 & \lambda x \\ \lambda\mu - \epsilon & \lambda x & \lambda^2(1 - w\epsilon) \end{bmatrix},$$

which is positive, by the Sylvester's criterion, if and only if all principal minors are positive. For all $\epsilon > 0$, the principal minor obtained by removing the first row and column is positive if and only if $w\epsilon < 1$ and its determinant

$$\lambda^2((1-2wx^2)\epsilon - w\epsilon^2)/2 \;=\; \lambda^2\epsilon((1-2wx^2) - w\epsilon)/2$$

is positive. This is clearly the case if $\epsilon < \min\{1, w^{-1}, (1-2wx^2)/w\} := \epsilon_1$.

The principal minor obtained by removing the second row and column, namely

$$\begin{bmatrix} \mu^2 + r^4(1-\epsilon) & \lambda\mu - \epsilon \\ \lambda\mu - \epsilon & \lambda^2(1-w\epsilon) \end{bmatrix}$$

has positive diagonal elements for $0 < \epsilon < \epsilon_1$ and determinant

$$\lambda^2 r^4 + (2\lambda\mu - \lambda^2\mu^2 w - \lambda^2 r^4(1+w))\epsilon + \lambda^2 r^4 w\epsilon^2$$

which is clearly strictly positive for all $0 < \epsilon < \epsilon_2$ for some $\epsilon_2 < \epsilon_1$. Finally, the principal minor obtained by removing the third row and column, namely

$$\begin{bmatrix} \mu^2 + r^4(1-\epsilon) & \mu x \\ \mu x & x^2 + \epsilon/2 \end{bmatrix}$$

which has positive diagonal elements for $\epsilon < 1$, has determinant

$$r^4x^2 + (\mu^2 + r^4 - 2r^4x^2)\epsilon/2 - r^4\epsilon^2/2\,.$$

This is clearly positive for all ϵ small enough if $x \neq 0$ because it tends to $r^4x^2 \neq 0$ but also for $x = 0$ since, in this case it is equal to $\epsilon(\mu^2 + r^4 - r^4\epsilon/2)$. This completes the proof. □

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