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The $\mathcal{P}$-Hausdorff, $\mathcal{P}$-regular and $\mathcal{P}$-normal ideal spaces

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Abstract:

We introduce and study new extensions of some separation axioms to ideal topological spaces, which we have called $\mathcal{P}$-Hausdorff, $\mathcal{P}$-regular and $\mathcal{P}$-normal. These extensions are quite natural and represent a good improvement with respect to other extensions that have recently occurred, in which a level of separation that can be considered acceptable is not perceived.

Keywords: Separation in ideal topological spaces; Hausdorff modulo $\mathcal{I}$; $\mathcal{I}$-Hausdorff; $\mathcal{I}$-regular; $\mathcal{I}$ normal.

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1. Introduction and preliminaries

Several extensions of separation axioms to ideal topological spaces have been considered by authors s uch as Dontchev, Hamlett-Jancovic, Renuka Devi-Sivaraj and Suriyakala-Vembu. In some cases these concepts are quite simple and natural, but these leave the distaste of not offering an adequate level of separation, neither between points nor between closed sets. In this work we propose to correct this deficiency, showing that the objects of interest really can be separated in some good way. More precisely, we will define and study the $\mathcal{P}$-Hausdorff, the $\mathcal{P}$-regular and the $\mathcal{P}$-normal spaces. We will also establish relationships between these new concepts and other separation axioms for ideal topological spaces, previously introduced.

An ideal $\mathcal{I}$ in a set X is a subset of $\mathcal{P}(X)$, the power set of X, such that:
(i) if $A \subseteq B \subseteq X$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$, and
(ii) if $\{A, B\} \subseteq \mathcal{I}$ then $A \cup B \in \mathcal{I}$.

Some useful ideals in X are: $(i)\mathcal{P}(A)$, where $A \subseteq X$, $(ii)\mathcal{I}_f(X)$, the ideal of all finite subsets of X, $(iii)\mathcal{I}_c(X)$, the ideal of all countable subsets of X and $(iv)\mathcal{I}_n(X)$, the ideal of all nowhere dense sets in a topological space (X, τ). If $\mathcal{I}$ is an ideal in X and if $f : X \to Y$ is a function, then the set $f(\mathcal{I}) = \{f(I) : I \in \mathcal{I}\}$ is an ideal in Y [5]. Furthermore, if $\mathcal{J}$ is an ideal in Y and if $f : X \to Y$ is an one-one function, then the set $f^{-1}(\mathcal{J}) = \{f^{-1}(J) : J \in \mathcal{J}\}$ is an ideal in X [5].

If (X, τ) is a topological space and $\mathcal{I}$ is an ideal in X, then $(X, \tau, \mathcal{I})$ is called an ideal space. If (X, τ) is a topological space and $A \subseteq X$ then the closure and the interior of A are denoted by $\overline{A}$ (or $adh_\tau(A)$, or $adh(A)$) and $\overset{0}{A}$ (or $int_\tau(A)$, or $int(A)$), respectively. Note that the set $\{U \backslash V : \{U, V\} \subseteq \tau\}$ is a base for a topology $\tau^\square$ in X, finer than τ.

Given an ideal space $(X, \tau, \mathcal{I})$ and a set $A \subseteq X$, we denote by $A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I}$, for every $U \in \tau$with $x \in U\}$, written simply as A^* when there is no chance for confusion. It is clear that $A^* \subseteq \overline{A}$. A Kuratowski closure operator [11] for a topology $\tau^*(\mathcal{I})$, finer than τ, is defined by $Cl^*(A) = A \cup A^*$, for all $A \subseteq X$. When there is no chance for confusion $\tau^*(\mathcal{I})$ is denoted by τ^*. The topology τ^* has as a base $\beta(\tau, \mathcal{I}) = \{V \backslash I : V \in \tau$ and $I \in \mathcal{I}\}$ [3]. In 1992 Hamlett and Jancovic introduced the notion of $\mathcal{I}$-open sets. If $(X, \tau, \mathcal{I})$ is an ideal space and $A \subseteq X$, A is defined to be $\mathcal{I}$-open [2] if $A \subseteq int_\tau(A^*)$.

If $\mathcal{I}$ is an ideal in X and $\mathcal{J}$ is an ideal in Y, then $\mathcal{I} \otimes \mathcal{J}$ [6] is the set of all $D \subseteq X \times Y$ such that there exist $I \in \mathcal{I}$, $A \subseteq X$, $J \in \mathcal{J}$ and $B \subseteq Y$, with $D \subseteq (A \times J) \cup (I \times B)$. It is shown in [6] that $\mathcal{I} \otimes \mathcal{J}$ is an ideal in $X \times Y$.

If $\{X_i : i \in \Lambda\}$ is a collection of sets and if $\mathcal{I}_i$ is an ideal in X_i, for each $i \in \Lambda$, we will denote by $\bigotimes\limits_{i\in\Lambda} \mathcal{I}_i$ the set of all $A \subseteq \prod\limits_{i\in\Lambda} X_i$ such that there exists a finite $\Lambda_0 \subseteq \Lambda$ with $A \subseteq \bigcup\limits_{i\in\Lambda_0} p_i^{-1}(I_i)$, for some $I_i \in \mathcal{I}_i$, for each $i \in \Lambda_0$ [7]. Here p_i represents the i-th projection. It is very simple to prove that $\bigotimes\limits_{i\in\Lambda} \mathcal{I}_i$ is an ideal in $\prod\limits_{i\in\Lambda} X_i$.

The symbol $\square$ is used to indicate the end of a proof.

2. $\mathcal{P}$-Hausdorff ideal spaces

Previous and different versions of the T_2 axiom for ideal spaces have been considered by Dontchev and Sivaraj-Renuka Devi. In this section we introduce and study the $\mathcal{P}$-Hausdorff spaces, a strong form of Hausdorff modulo $\mathcal{I}$ spaces. The $\mathcal{P}$-Hausdorffness and $\mathcal{I}$-Hausdorffness turn out to be independent concepts. We recall that an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-Hausdorff [1] if for each $\{a, b\} \subseteq X$, with $a \neq b$, there are disjoint $\mathcal{I}$-open sets U and V, such that $a \in U$ and $b \in V$. On the other hand, $(X, \tau, \mathcal{I})$ is defined to be Hausdorff modulo $\mathcal{I}$ [9] if for each $\{a, b\} \subseteq X$, with $a \neq b$, there is a $\{U, V\} \subseteq \tau$, such that $a \in U$, $b \in V$ and $U \cap V \in \mathcal{I}$. This last version, although interesting, does not seem entirely satisfactory, given that the case may arise in which, for some a and b, we have $\{a, b\} \subseteq U$ or $\{a, b\} \subseteq V$. Some interesting additional properties of such spaces were presented in [7].

Definition 2.1 The ideal space $(X, \tau, \mathcal{I})$ is defined to be $\mathcal{P}$-Hausdorff if for each $\{a, b\} \subseteq X$, with $a \neq b$, there is a $\{U, V\} \subseteq \tau$ such that $a \in U \backslash V, b \in V \backslash U$ and $U \cap V \in \mathcal{I}$. A set $A \subseteq X$ is said to be $\mathcal{P}$-Hausdorff if $(A, \tau_A, \mathcal{I}_A)$ is $\mathcal{P}$-Hausdorff, where $\mathcal{I}_A = \{I \cap A : I \in \mathcal{I}\}$ and $\tau_A = \{U \cap A : U \in \tau\}$.

It is noted that $(X, \tau, \mathcal{I}_f(X))$ is $\mathcal{P}$-Hausdorff if and only if (X, τ) is Hausdorff, since $\mathcal{P}$-Hausdorff $\rightarrow T_1$. It is also evident that Hausdorff $\rightarrow \mathcal{P}$-Hausdorff, and that if $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-Hausdorff, then $(X, \tau, \mathcal{I})$ is Hausdorff modulo $\mathcal{I}$, and (X, τ^*) and $(X, \tau^\square)$ are Hausdorff spaces.

Example 2.2
1) If $\mu = \{V \subseteq \mathbf{R} : \mathbf{R} \backslash V \text{ is finite}\} \cup \{\emptyset\}$ and if $\mathcal{I} = \mathcal{I}_c(\mathbf{R})$, then $(\mathbf{R}, \mu)$ is a T_1 space, but $(\mathbf{R}, \mu, \mathcal{I})$ is not a $\mathcal{P}$-Hausdorff space.

2) In the set $\mathbf{R}$ consider the topology β that consists of all $V \subseteq \mathbf{R}$ that satisfy that, for all $a \in V$, there is a $r > a$ such that $\{a\} \cup (r, +\infty) \subseteq V$. Let $\mathcal{I}$ be the ideal of all subsets of $\mathbf{R}$ that are bounded below. If $\{a, b\} \subseteq \mathbf{R}$ and $a < b$, then $\{\{a\} \cup (b+1, +\infty), \{b\} \cup (b+1, +\infty)\} \subseteq \beta$ and $[\{a\} \cup (b+1, +\infty)] \cap [\{b\} \cup (b+1, +\infty)] = (b+1, +\infty) \in \mathcal{I}$. Thus $(\mathbf{R}, \beta, \mathcal{I})$ is a $\mathcal{P}$-Hausdorff space.
3) (An $\mathcal{I}$-Hausdorff but not $\mathcal{P}$-Hausdorff space) If β is the topology $\{\emptyset, \mathbf{R}\} \cup \{(a, +\infty) : a \in \mathbf{R}\}$ and if $\mathcal{I} = \mathcal{I}_f(\mathbf{R})$, then $(\mathbf{R}, \beta, \mathcal{I})$ is $\mathcal{I}$-Hausdorff [1]. However, it is clear that this space is not $\mathcal{P}$-Hausdorff.
4) (A Hausdorff modulo $\mathcal{I}$ but not $\mathcal{P}$-Hausdorff space) If $\beta = \{\emptyset, \mathbf{R}\} \cup \{(a, +\infty) : a \in \mathbf{R}\}$ and if $\mathcal{I}$ is the ideal of all subsets of $\mathbf{R}$ that are bounded below, then $(\mathbf{R}, \beta, \mathcal{I})$ is Hausdorff modulo $\mathcal{I}$, because if $a < b$ and if $U = V = (a-1, +\infty)$ then $a \in U$, $b \in V$, and $U \cap V \in \mathcal{I}$. But it is evident that $(\mathbf{R}, \beta, \mathcal{I})$ is not $\mathcal{P}$-Hausdorff.
5) (A $\mathcal{P}$-Hausdorff but not $\mathcal{I}$-Hausdorff space) If $X = \{a, b\}$, $\tau = \mathcal{P}(\{a, b\}) = \mathcal{I}$, then $(X, \tau, \mathcal{I})$ is not $\mathcal{I}$-Hausdorff [1]. However $(X, \tau, \mathcal{I})$ is a $\mathcal{P}$-Hausdorff space, since (X, τ) is Hausdorff.

Observe that $\mathcal{P}$-Hausdorff and $\mathcal{I}$-Hausdorff are independent concepts.

Theorem 2.3 If $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-Hausdorff and if $A \subseteq X$, then A is $\mathcal{P}$-Hausdorff.

Proof. If $\{a, b\} \subseteq A$ and $a \neq b$, there exists $\{U, V\} \subseteq \tau$ such that $a \in U \backslash V, b \in V \backslash U$ and $U \cap V \in \mathcal{I}$. Thus $a \in (U \cap A) \setminus (V \cap A)$, $b \in (V \cap A) \setminus (U \cap A)$ and $(V \cap A) \cap (U \cap A) = (U \cap V) \cap A \in \mathcal{I}_A$. □

In the three following theorems we consider the products and sums of $\mathcal{P}$-Hausdorff spaces.

Theorem 2.4 If $(X, \tau, \mathcal{I})$ and $(Y, \beta, \mathcal{L})$ are $\mathcal{P}$-Hausdorff, then the space $(X \times Y, \tau \times \beta, \mathcal{I} \otimes \mathcal{L})$ is $\mathcal{P}$-Hausdorff.

Proof. Let $\{(a, b), (c, d)\} \subseteq X \times Y$ be such that $(a, b) \neq (c, d)$. If $a \neq c$ then there is a $\{U_1, U_2\} \subseteq \tau$ such that $\{a, c\} \cap U_1 = \{a\}$, $\{a, c\} \cap U_2 = \{c\}$ and $U_1 \cap U_2 \in \mathcal{I}$. So $\{(a, b), (c, d)\} \cap (U_1 \times Y) = \{(a, b)\}$, $\{(a, b), (c, d)\} \cap (U_2 \times Y) = \{(c, d)\}$ and $(U_1 \times Y) \cap (U_2 \times Y) = (U_1 \cap U_2) \times Y \in \mathcal{I} \otimes \mathcal{L}$. Similarly, if $b \neq d$ we can find a $\{W_1, W_2\} \subseteq \tau \times \beta$ such that $\{(a, b), (c, d)\} \cap W_1 = \{(a, b)\}$, $\{(a, b), (c, d)\} \cap W_2 = \{(c, d)\}$ and $W_1 \cap W_2 \in \mathcal{I} \otimes \mathcal{L}$. Hence $(X \times Y, \tau \times \beta, \mathcal{I} \otimes \mathcal{L})$ is $\mathcal{P}$-Hausdorff. □

Theorem 2.5 If $\{(X_\alpha, \tau_\alpha, \mathcal{I}_\alpha) : \alpha \in \Lambda\}$ is a collection of $\mathcal{P}$-Hausdorff spaces, then $\left(\prod_{\alpha\in\Lambda} X_\alpha, \prod_{\alpha\in\Lambda} \tau_\alpha, \bigotimes_{\alpha\in\Lambda} \mathcal{I}_\alpha\right)$ is a $\mathcal{P}$-Hausdorff space.

Proof. If $\{a = (a_\alpha)_{\alpha\in\Lambda}, b = (b_\alpha)_{\alpha\in\Lambda}\} \subseteq \prod_{\alpha\in\Lambda} X_\alpha$, with $a \neq b$, there exists $\delta \in \Lambda$ such that $a_\delta \neq b_\delta$. Since $(X_\delta, \tau_\delta, \mathcal{I}_\delta)$ is $\mathcal{P}$-Hausdorff, there is a $\{U_\delta, V_\delta\} \subseteq \tau_\delta$ such that $a_\delta \in U_\delta \backslash V_\delta$, $b_\delta \in V_\delta \backslash U_\delta$ and $U_\delta \cap V_\delta \in \mathcal{I}_\delta$. If p_δ represents the δ-th proyection then $a \in p_\delta^{-1}(U_\delta) \backslash p_\delta^{-1}(V_\delta)$, $b \in p_\delta^{-1}(V_\delta) \backslash p_\delta^{-1}(U_\delta)$ and $p_\delta^{-1}(U_\delta) \cap p_\delta^{-1}(V_\delta) = p_\delta^{-1}(U_\delta \cap V_\delta) \in \bigotimes_{\alpha\in\Lambda} \mathcal{I}_\alpha$. □

If $\{X_i : i \in \Lambda\}$ is a collection of sets such that $X_i \cap X_j = \emptyset$, for each $\{i, j\} \subseteq \Lambda$ with $i \neq j$, and if $\mathcal{I}_i$ is an ideal in X_i, for each $i \in \Lambda$, we will denote by $\sum_{i\in\Lambda} \mathcal{I}_i$ the set $\left\{\sum_{i\in\Lambda} I_i : I_i \in \mathcal{I}_i, \text{ for each } i \in \Lambda\right\}$. It is clear that $\sum_{i\in\Lambda} \mathcal{I}_i$ is an ideal in $\sum_{i\in\Lambda} X_i$. On the other hand, if τ_i is a topology on X_i, for each $i \in \Lambda$, then the topology $\sum_{i\in\Lambda} \tau_i$ is the set $\left\{A \subseteq \sum_{i\in\Lambda} X_i : A \cap X_i \in \tau_i, \text{ for each } i \in \Lambda\right\}$.

Theorem 2.6 If $\{(X_\alpha, \tau_\alpha, \mathcal{I}_\alpha) : \alpha \in \Lambda\}$ is a collection of disjoint $\mathcal{P}$-Hausdorff spaces, then $\left(\sum_{\alpha\in\Lambda} X_\alpha, \sum_{\alpha\in\Lambda} \tau_\alpha, \sum_{\alpha\in\Lambda} \mathcal{I}_\alpha\right)$ is a $\mathcal{P}$-Hausdorff space.

Proof. Suppose that $\{a, b\} \subseteq \sum_{\alpha\in\Lambda} X_\alpha$, with $a \neq b$. We have that:
(a) If there is a $\delta \in \Lambda$ such that $\{a, b\} \subseteq X_\delta$, then there exists $\{U_\delta, V_\delta\} \subseteq \tau_\delta \subseteq \sum_{\alpha\in\Lambda} \tau_\alpha$ such that $a \in U_\delta \backslash V_\delta$, $b \in V_\delta \backslash U_\delta$ and $U_\delta \cap V_\delta \in \mathcal{I}_\delta \subseteq \sum_{\alpha\in\Lambda} \mathcal{I}_\alpha$.
(b) If there is a $\{\delta, \sigma\} \subseteq \Lambda$ such that $\delta \neq \sigma$, $a \in X_\delta$ and $b \in X_\sigma$, then $\{X_\delta, X_\sigma\} \subseteq \sum_{\alpha\in\Lambda} \tau_\alpha$, $X_\delta \cap X_\sigma = \emptyset \in \sum_{\alpha\in\Lambda} \mathcal{I}_\alpha$, $a \in X_\delta \backslash X_\sigma$ and $b \in X_\sigma \backslash X_\delta$. □

Below we present some functional properties about $\mathcal{P}$-Hausdorff spaces.

Theorem 2.7
1) If $f : X \to Y$ is an one-one function and if $(Y, \beta, \mathcal{L})$ is a $\mathcal{P}$-Hausdorff space, then $\left(X, f^{-1}(\beta), f^{-1}(\mathcal{L})\right)$ is $\mathcal{P}$-Hausdorff. 2) If $f : (X, \tau) \to (Y, \beta)$ is a biyective and open function, $\mathcal{I}$ is an ideal in X, and if $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-Hausdorff, then $(Y, \beta, f(\mathcal{I}))$ is $\mathcal{P}$-Hausdorff.

Proof.
1) If $\{a, b\} \subseteq X$, with $a \neq b$, there is a $\{V, W\} \subseteq \beta$ such that $f(a) \in V \backslash W$,

$f(b) \in W \backslash V$ and $V \cap W \in \mathcal{L}$. This implies that $a \in f^{-1}(V) \backslash f^{-1}(W)$ and $b \in f^{-1}(W) \backslash f^{-1}(V)$. Moreover $f^{-1}(V) \cap f^{-1}(W) = f^{-1}(V \cap W) \in f^{-1}(\mathcal{L})$ and $\{f^{-1}(V), f^{-1}(W)\} \subseteq f^{-1}(\beta)$.
2) If $\{a, b\} \subseteq X$ and $f(a) \neq f(b)$, there is a $\{U, V\} \subseteq \tau$ such that $a \in U \backslash V$, $b \in V \backslash U$ and $U \cap V \in \mathcal{I}$. Hence $f(a) \in f(U) \backslash f(V)$, $f(b) \in f(V) \backslash f(U)$ and $f(U) \cap f(V) = f(U \cap V) \in f(\mathcal{I})$. □

Theorem 2.8 Suppose that $f : (X, \tau) \to (Y, \beta)$ and $g : (X, \tau) \to (Y, \beta)$ are continuous functions, that $(Y, \beta, \mathcal{L})$ is a $\mathcal{P}$-Hausdorff space, and that $\mathcal{L}$ is an ideal in $Y \backslash f(X)$. Then the set $A = \{x \in X : f(x) = g(x)\}$ is closed in (X, τ).

Proof. If $b \in X \backslash A$ then there is a $\{V, W\} \subseteq \beta$ such that $f(b) \in V \backslash W$, $g(b) \in W \backslash V$ and $V \cap W \in \mathcal{L}$. Hence $b \in [f^{-1}(V) \backslash f^{-1}(W)] \cap [g^{-1}(W) \backslash g^{-1}(V)] \subseteq f^{-1}(V) \cap g^{-1}(W)$.
Suppose that there is a $z \in [f^{-1}(V) \cap g^{-1}(W)] \cap A$. Thus $f(z) = g(z) \in V \cap W$, and so $\{f(z)\} \in \mathcal{L}$, absurd. Then $f^{-1}(V) \cap g^{-1}(W) \subseteq X \backslash A$. □

Now we present two properties of separation for compact subsets in $\mathcal{P}$-Hausdorff spaces.

Theorem 2.9 If $(X, \tau, \mathcal{I})$ is a $\mathcal{P}$-Hausdorff space, $K \subseteq X \backslash \bigcup_{I \in \mathcal{I}} I$ is compact and if $a \in X \backslash K$, then there exists $\{U, V\} \subseteq \tau$ such that $(\{a\} \cup K) \cap U = \{a\}$, $(\{a\} \cup K) \cap V = K$ and $U \cap V \in \mathcal{I}$.

Proof. If $x \in K$ then there is a $\{U_x, V_x\} \subseteq \tau$ such that $\{a, x\} \cap U_x = \{a\}$, $\{a, x\} \cap V_x = \{x\}$ and $U_x \cap V_x \in \mathcal{I}$. There exists a finite $K_0 \subseteq K$ such that $K \subseteq V = \bigcup_{x \in K_0} V_x$. Let $U = \bigcap_{x \in K_0} U_x$ be. It is clear that $U \cap V \in \mathcal{I}$. Given that $a \notin V_x$, for each $x \in K$, then $a \notin V$ and so $(\{a\} \cup K) \cap V = K$. Now, if there is a $z \in U \cap K$ then $z \in U_{x_1} \cap V_{x_1}$, for some $x_1 \in K_0$. This implies that $\{z\} \in \mathcal{I}$, but it is absurd since $K \subseteq X \backslash \bigcup_{I \in \mathcal{I}} I$. Consequently $(\{a\} \cup K) \cap U = \{a\}$. □

Theorem 2.10 If $(X, \tau, \mathcal{I})$ is a $\mathcal{P}$-Hausdorff space and if $K \subseteq X \backslash \bigcup_{I \in \mathcal{I}} I$ and $L \subseteq X \backslash \bigcup_{I \in \mathcal{I}} I$ are disjoint and compact, then there exists $\{U, V\} \subseteq \tau$ such that $(L \cup K) \cap U = L$, $(L \cup K) \cap V = K$ and $U \cap V \in \mathcal{I}$.

Proof. If $x \in K$ then there is a $\{U_x, V_x\} \subseteq \tau$ such that $(\{x\} \cup L) \cap U_x = L$, $(\{x\} \cup L) \cap V_x = \{x\}$ and $U_x \cap V_x \in \mathcal{I}$, by Theorem 2.9. There exists a finite $K_0 \subseteq K$ such that $K \subseteq V = \bigcup_{x \in K_0} V_x$. Let $U = \bigcap_{x \in K_0} U_x$ be. It is clear that $U \cap V \in \mathcal{I}$ and $L \subseteq U$. Suppose that there is a $y \in V \cap L$. Then there exists $x_0 \in K_0$ such that $y \in V_{x_0} \cap U_{x_0}$. This implies that $\{y\} \in \mathcal{I}$, but it is impossible since $L \subseteq X \backslash \bigcup_{I \in \mathcal{I}} I$. Hence $(K \cup L) \cap V = K$. Similarly we obtain that $(K \cup L) \cap U = L$. □

The result that follows is related to the convergence of sequences in $\mathcal{P}$-Hausdorff spaces.

Theorem 2.11 If $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-Hausdorff and $\{x_n\}$ is a succession in X such that there is no a positive integer M such that $\{x_n : n \geq M\} \in \mathcal{I}$, then if $\{x_n\}$ converge to a and b, we have that $a = b$.

Proof. If $a \neq b$ then there exists $\{U, V\} \subseteq \tau$ such that $a \in U \backslash V$, $b \in V \backslash U$ and $U \cap V \in \mathcal{I}$. Now, since $\{x_n\}$ converge to a and b, there is a $M \in \mathbf{Z}^+$ such that $\{x_n : n \geq M\} \subseteq U \cap V$. This implies that $\{x_n : n \geq M\} \in \mathcal{I}$, absurd. □

3. $\mathcal{P}$-regular ideal spaces

In 1994 Jancovic and Hamlett define the $\mathcal{I}$-regular spaces in [4], and in 2016 Suriyakala and Vembu define the $\mathcal{J}$ -regular spaces in [10]. In this section we introduce and study the $\mathcal{P}$-regular spaces, a weak form of the $\mathcal{J}$-regular spaces. The concepts of $\mathcal{P}$-regularity and $\mathcal{I}$-regularity turn out to be independent. An ideal space $(X, \tau, \mathcal{I})$ is said to be: (i) $\mathcal{I}$-regular if for every closed set $F \subseteq X$ and $x \in X \backslash F$ there exist disjoint open sets U and V such that $x \in U$ and $F \backslash V \in \mathcal{I}$, and (ii) $\mathcal{J}$-regular if (X, τ) is T_1 and if, for every closed set $F \subseteq X$ and $x \in X \backslash F$, there is a $\{U, V\} \subseteq \tau$ such that $x \in U$, $F \subseteq V$ and $U \cap V \in \mathcal{I}$. This last definition is a bit disappointing, because the case may arise in which, for some a and F, we have $F \cup \{a\} \subseteq U$ or $F \cup \{a\} \subseteq V$. Several interesting additional properties of $\mathcal{J}$-regular spaces were presented in [7].

Definition 3.1 The ideal space $(X, \tau, \mathcal{I})$ is defined to be $\mathcal{P}$-regular if for each closed set $F \subseteq X$ and each $a \in X \backslash F$, there is a $\{U, V\} \subseteq \tau$ such that $(\{a\} \cup F) \cap U = \{a\}$, $(\{a\} \cup F) \cap V = F$ and $U \cap V \in \mathcal{I}$. If $A \subseteq X$ then A is $\mathcal{P}$-regular if $(A, \tau_A, \mathcal{I}_A)$ is a $\mathcal{P}$-regular space.
It is observed that Regular $\rightarrow$ $\mathcal{P}$-regular and that $(X, \tau, \{\emptyset\})$ is $\mathcal{P}$-regular if and only if (X, τ) is regular. Now, if $(X, \tau, \mathcal{I})$ is $\mathcal{J}$-regular then $(X, \tau, \mathcal{I})$

is $\mathcal{P}$-regular, because if F is a closed set and if $a \in X\backslash F$ then there exists $\{U, V\} \subseteq \tau$ such that $a \in U$, $F \subseteq V$ and $U \cap V \in \mathcal{I}$. Then $(F \cup \{a\}) \cap (U\backslash F) = \{a\}$, $(F \cup \{a\}) \cap (V\backslash \{a\}) = F$, $\{U\backslash F, V\backslash \{a\}\} \subseteq \tau$ and $(U\backslash F) \cap (V\backslash \{a\}) \in \mathcal{I}$. In the next example we show that the reciprocal affirmation is, in general, false.
Then $T_3 \to \mathcal{J}$-regular $\to \mathcal{P}$-regular.

Example 3.2 1) (A $\mathcal{P}$-regular but not $\mathcal{I}$-regular space) In the set $\mathbf{R}$ consider the topology β that consists of all $V \subseteq \mathbf{R}$ that satisfy that, for all $a \in V$, there is a $r > a$ such that $\{a\} \cup (r, +\infty) \subseteq V$. The space $(\mathbf{R}, \beta)$ is not regular since there are no disjoint open U and V such that $1 \in U$ and $(-\infty, 0] \subseteq V$. Let $\mathcal{I}$ be the ideal of all subsets of $\mathbf{R}$ that are bounded below. Suppose that F is closed in $(\mathbf{R}, \beta)$ and that $a \in \mathbf{R}\backslash F$. There is a $r > a$ such that $\{a\} \cup (r, +\infty) \subseteq \mathbf{R}\backslash F$. Thus $(\{a\} \cup F) \cap (\{a\} \cup (r, +\infty)) = \{a\}$, $(\{a\} \cup F) \cap (\mathbf{R}\backslash \{a\}) = F$, $\{\{a\} \cup (r, +\infty), \mathbf{R}\backslash \{a\}\} \subseteq \beta$ and $(\{a\} \cup (r, +\infty)) \cap (\mathbf{R}\backslash \{a\}) = (r, +\infty) \in \mathcal{I}$. Thus $(\mathbf{R}, \beta, \mathcal{I})$ is a $\mathcal{P}$-regular space. However this space is not $\mathcal{I}$-regular because the set $F = (-\infty, 0]$ is closed, but there are no disjoint open sets U and V such that $1 \in U$ and $F\backslash V \in \mathcal{I}$.
2) In $\mathbf{R}$ consider the topology γ in which the open neighborhoods of a real $r \neq 0$ are the usual ones, and the open neighborhoods of 0 are of the form $U\backslash F$, where $F = \{1/n : n \in \mathbf{Z}^+\}$ and U is an usual open neighborhood of 0. It is well known that F is closed in $(\mathbf{R}, \gamma)$ and that the space (X, γ) is Hausdorff but not regular. Let $\mathcal{I} = \mathcal{P}(\mathbf{Q})$ be. If $\{U, V\} \subseteq \gamma$, $0 \in U$ and $F \subseteq V$, then is clear that $U \cap V \cap (\mathbf{R}\backslash\mathbf{Q}) \neq \emptyset$, and so $U \cap V \notin \mathcal{I}$. Hence $(X, \gamma, \mathcal{I})$ is not a $\mathcal{P}$-regular space. However $(X, \gamma, \mathcal{I})$ is $\mathcal{P}$-Hausdorff.
3) (An $\mathcal{I}$-regular but not $\mathcal{P}$-regular space) If we consider $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$ then it's very easy to check that $(X, \tau, \mathcal{I})$ is an $\mathcal{I}$-regular space. However $(X, \tau, \mathcal{I})$ is not a $\mathcal{P}$-regular space, because the set $F = \{a, b, d\}$ is closed but there is no an open set U such that $[F \cup \{c\}] \cap U = F$.
4) (A $\mathcal{P}$-regular and not $\mathcal{J}$-regular space) If $X = \{a, b, c\}$ and $\lambda = \{\emptyset, X, \{a, b\}, \{c\}\}$, then $(X, \lambda, \{\emptyset\})$ is a $\mathcal{P}$-regular and not $\mathcal{J}$-regular space.

These examples show that $\mathcal{I}$-regular and $\mathcal{P}$-regular are independent concepts.
The following theorem shows a characterization of $\mathcal{P}$-regular spaces.

Theorem 3.3 1) If $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-regular and $A \subseteq X$, then A is $\mathcal{P}$-regular.
2) The ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-regular if and only if, for each $U \in \tau$ and

each $x \in U$, there are $V \in \tau$ and a closed set $G \subseteq X$ such that $x \in V \subseteq U$, $x \in G \subseteq U$ and $V \backslash G \in \mathcal{I}$.

Proof. 1) Suppose that G is closed in (A, τ_A) and $b \in A \backslash G$. Since $G = \overline{G} \cap A$ then $b \notin \overline{G}$. There is a $\{U, V\} \subseteq \tau$ such that $\left(\{b\} \cup \overline{G}\right) \cap U = \{b\}$, $\left(\{b\} \cup \overline{G}\right) \cap V = \overline{G}$ and $U \cap V \in \mathcal{I}$. Then $(U \cap A) \cap (V \cap A) = (U \cap V) \cap A \in \mathcal{I}_A$, $(\{b\} \cup G) \cap (U \cap A) = \{b\}$, and $(\{b\} \cup G) \cap (V \cap A) = \left[\{b\} \cup \left(\overline{G} \cap A\right)\right] \cap (V \cap A) = \left[\left(\{b\} \cup \overline{G}\right) \cap A\right] \cap (V \cap A) = \left[\left(\{b\} \cup \overline{G}\right) \cap V\right] \cap A = \overline{G} \cap A = G$.
2) $(\rightarrow)$ If $U \in \tau$ and $x \in U$, there is a $\{V, W\} \subseteq \tau$ such that $(\{x\} \cup (X \backslash U)) \cap V = \{x\}$, $(\{x\} \cup (X \backslash U)) \cap W = X \backslash U$ and $V \cap W \in \mathcal{I}$. Hence $x \in V \subseteq U$, $x \in X \backslash W \subseteq U$ and $V \backslash (X \backslash W) \in \mathcal{I}$. $(\leftarrow)$ Suppose that F is a closed set and that $x \in X \backslash F$. There are a closed set $G \subseteq X$ and a $V \in \tau$ such that $x \in V \subseteq X \backslash F$, $x \in G \subseteq X \backslash F$ and $V \backslash G \in \mathcal{I}$. Thus $(\{x\} \cup F) \cap V = \{x\}$, $(\{x\} \cup F) \cap (X \backslash G) = F$ and $V \cap (X \backslash G) \in \mathcal{I}$. $\square$

If $(X, \tau, \mathcal{I})$ is an ideal space, the ideal $\overline{\mathcal{I}}$ [6] is defined as the set of all $A \subseteq X$ such that there is a $I \in \mathcal{I}$ with $A \subseteq \overline{I}$. It is clear that, for all $A \subseteq X$, $A \in \overline{\mathcal{I}}$ if and only if $\overline{A} \in \overline{\mathcal{I}}$.

Definition 3.4 If $(X, \tau, \mathcal{I})$ is an ideal space, $\mathcal{I}$ is said to be closed in (X, τ) if $\overline{I} \in \mathcal{I}$, for each $I \in \mathcal{I}$.

It is observed that $\mathcal{I}$ is closed in (X, τ) if and only if $\overline{\mathcal{I}} = \mathcal{I}$.

Example 3.5

1) If (X, τ) is a topological space and $A \subseteq X$ then the set $\mathcal{I} = \left\{B \subseteq X : B \subseteq \overline{A}\right\}$ is a closed ideal in (X, τ).

2) If (X, τ) is a topological space then $\mathcal{I}_n$ is a closed ideal in (X, τ).

Theorem 3.6

1) If $\mathcal{I}$ and $\mathcal{J}$ are closed ideals in (X, τ), then $\mathcal{I} \vee \mathcal{J}$ is closed in (X, τ).
b) If $\{\mathcal{I}_\alpha : \alpha \in \Delta\}$ is a collection of closed ideals in (X, τ), then $\bigcap_{\alpha \in \Delta} \mathcal{I}_\alpha$ is closed in (X, τ).
2) If $\mathcal{I}$ and $\mathcal{J}$ are closed ideals in (X, τ) and (Y, β), respectively, then $\mathcal{I} \otimes \mathcal{J}$ is closed in $(X \times Y, \tau \times \beta)$.
3) If $\mathcal{I}_\alpha$ is a closed ideal in (X_α, τ_α), for each $\alpha \in \Delta$, then $\bigotimes_{\alpha \in \Delta} \mathcal{I}_\alpha$ is closed in $\left(\prod_{\alpha \in \Delta} X_\alpha, \prod_{\alpha \in \Delta} \tau_\alpha\right)$.

4)$Iff:(X,\tau)\rightarrow(Y,\beta)$ is a closed function and if $\mathcal{I}$ is a closed ideal in (X,τ), then $f(\mathcal{I})$ is closed in (Y,β).
5) If $f:(X,\tau)\rightarrow(Y,\beta)$ is an one-one continuous function, and if $\mathcal{J}$ is a closed ideal in (Y,β), then $f^{-1}(\mathcal{J})$ is closed in (X,τ).

Proof.
1) It is very simple.
2) Suppose that $W\in\mathcal{I}\otimes\mathcal{J}$. There are $A\subseteq X$, $B\subseteq Y$, $I\in\mathcal{I}$ and $J\in\mathcal{J}$ such that $W\subseteq(I\times B)\cup(A\times J)$. Thus $\overline{W}\subseteq\left(\overline{I}\times\overline{B}\right)\cup\left(\overline{A}\times\overline{J}\right)$, and so $\overline{W}\in\mathcal{I}\otimes\mathcal{J}$, because $\overline{I}\in\mathcal{I}$ and $\overline{J}\in\mathcal{J}$.
3) If $W\in\bigotimes_{\alpha\in\Delta}\mathcal{I}_\alpha$ then there is a finite $\Delta_0\subseteq\Delta$ such that, for each $\alpha\in\Delta_0$, there exists $I_\alpha\in\mathcal{I}_\alpha$ with $W\subseteq\bigcap_{\alpha\in\Delta_0}p_\alpha^{-1}(I_\alpha)$. Hence $\overline{W}\subseteq\overline{\bigcap_{\alpha\in\Delta_0}p_\alpha^{-1}(I_\alpha)}\subseteq\bigcap_{\alpha\in\Delta_0}\overline{p_\alpha^{-1}(I_\alpha)}\subseteq\bigcap_{\alpha\in\Delta_0}p_\alpha^{-1}\left(\overline{I_\alpha}\right)$, and so $\overline{W}\in\bigotimes_{\alpha\in\Delta}\mathcal{I}_\alpha$, given that $\overline{I_\alpha}\in\mathcal{I}_\alpha$, for each $\alpha\in\Delta_0$.
4) If $I\in\mathcal{I}$ then $\overline{f(I)}\subseteq f\left(\overline{I}\right)$. This implies that $\overline{f(I)}\in f(\mathcal{I})$, given that $\overline{I}\in\mathcal{I}$.
5) Suppose that $J\in\mathcal{J}$. Since $\overline{f^{-1}(J)}\subseteq f^{-1}\left(\overline{J}\right)$ and $\overline{J}\in\mathcal{J}$, we obtain that $\overline{f^{-1}(J)}\in f^{-1}(\mathcal{J})$. □

The following theorem shows a characterization of $\mathcal{P}$-regular spaces, in the case of closed ideals.

Theorem 3.7 If $\mathcal{I}$ is a closed ideal in (X,τ) then:
1) $(X,\tau,\mathcal{I})$ is $\mathcal{P}$-regular if and only if for each $U\in\tau$ and $x\in U$, there are $V\in\tau$ and a closed set $G\subseteq X$ such that $x\in V\subseteq U$, $x\in G\subseteq U$ and $\overline{V}\backslash G\in\mathcal{I}$.
2) If $(X,\tau,\mathcal{I})$ is $\mathcal{P}$-regular and $\{a,b\}\subseteq X$ then either $\overline{\{a\}}=\overline{\{b\}}$ or $\overline{\{a\}}\cap\overline{\{b\}}\in\mathcal{I}$.

Proof.
1) $(\rightarrow)$ Suppose that $U\in\tau$ and $x\in U$. There are $V\in\tau$, a closed set G and $I\in\mathcal{I}$ such that $x\in V\subseteq U$, $x\in G\subseteq U$ and $V\backslash G=I$, by Theorem 3.3. Then $V\subseteq G\cup I$ and so $\overline{V}\subseteq G\cup\overline{I}$. In this way $\overline{V}\backslash G\subseteq\overline{I}\in\mathcal{I}$ and so $\overline{V}\backslash G\in\mathcal{I}$. $(\leftarrow)$ This is immediate if we apply Theorem 3.3.
2) Suppose that $\overline{\{a\}}\neq\overline{\{b\}}$ and, without loss of generality, that $a\notin\overline{\{b\}}$. By part (1), there exist $V\in\tau$ and a closed set G such that $a\in V\subseteq X\backslash\overline{\{b\}}$, $a\in G\subseteq X\backslash\overline{\{b\}}$ and $\overline{V}\backslash G\in\mathcal{I}$. Thus $\overline{\{a\}}\cap\overline{\{b\}}\subseteq\overline{V}\cap(X\backslash G)\in\mathcal{I}$. □

The four theorems that follow show us something of the functional behavior of $\mathcal{P}$-regular spaces.

Theorem 3.8 If $f : (X, \tau) \to (Y, \beta)$ is a continuous, open, closed and sobreyective function and if $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-regular, then $(Y, \beta, f(\mathcal{I}))$ is $\mathcal{P}$-regular.

Proof. Suppose that $H \subseteq Y$ is closed and that $b = f(a) \in Y \backslash H$. Given that $a \in X \backslash f^{-1}(H)$ then there is a $\{U, V\} \subseteq \tau$ such that $U \cap V \in \mathcal{I}$, $[\{a\} \cup f^{-1}(H)] \cap U = \{a\}$ and $[\{a\} \cup f^{-1}(H)] \cap V = f^{-1}(H)$. So $b = f(a) \in f(U)$ and $H \subseteq Y \backslash [f(X \backslash V)]$. Since $a \in X \backslash V$ then $b \notin Y \backslash f(X \backslash V)$. On the other hand, given that $f^{-1}(H) \cap U = \emptyset$ we have that $f(U) \cap H = \emptyset$. Hence $[\{b\} \cup H] \cap f(U) = \{b\}$ and $[\{b\} \cup H] \cap [Y \backslash f(X \backslash V)] = H$. Now, let $I \in \mathcal{I}$ be such that $U \cap V = I$. Thus $U \subseteq (X \backslash V) \cup I$, $f(U) \subseteq f(X \backslash V) \cup f(I)$ and $f(U) \cap [Y \backslash f(X \backslash V)] \subseteq f(I) \in f(\mathcal{I})$. This implies that $f(U) \cap [Y \backslash f(X \backslash V)] \in f(\mathcal{I})$. □

Theorem 3.9 If $f : X \to Y$ is an one-one function and $(Y, \beta, \mathcal{J})$ is a $\mathcal{P}$-regular space, then $(X, f^{-1}(\beta), f^{-1}(\mathcal{J}))$ is $\mathcal{P}$-regular.

Proof. Suppose that $F \subseteq X$ is closed and that $a \in X \backslash F$. There exists a closed set $G \subseteq Y$ such that $F = f^{-1}(G)$. Since $f(a) \notin G$, there is a $\{U, V\} \subseteq \beta$ such that $[\{f(a)\} \cup G] \cap U = \{f(a)\}$, $[\{f(a)\} \cup G] \cap V = G$ and $U \cap V \in \mathcal{J}$. Then $a \in f^{-1}(U)$, $F = f^{-1}(G) \subseteq f^{-1}(V)$, $a \notin f^{-1}(V)$, $F \cap f^{-1}(U) = f^{-1}(G \cap U) = \emptyset$, $\{f^{-1}(U), f^{-1}(V)\} \subseteq f^{-1}(\beta)$ and $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) \in f^{-1}(\mathcal{J})$. □

Theorem 3.10 If $f : (X, \tau) \to (Y, \beta)$ is a continuous, closed and sobreyective function, $f^{-1}(\{y\})$ is compact, for all $y \in Y$, and if $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-regular, then $(Y, \beta, f(\mathcal{I}))$ is $\mathcal{P}$-regular.

Proof. Suppose that $K \subseteq Y$ is closed and $u \in Y \backslash K$. For every $x \in f^{-1}(\{u\})$ there is a $\{U_x, V_x\} \subseteq \tau$ such that $[\{x\} \cup f^{-1}(K)] \cap U_x = \{x\}$, $[\{x\} \cup f^{-1}(K)] \cap V_x = f^{-1}(K)$ and $U_x \cap V_x \in \mathcal{I}$. There exists a finite set $A \subseteq f^{-1}(\{u\})$ such that $f^{-1}(\{u\}) \subseteq U = \bigcup_{x \in A} U_x$. Let $V = \bigcap_{x \in A} V_x$ be. It is clear that $u \in Y \backslash f(X \backslash U)$ and $U \cap V \in \mathcal{I}$. Given that $f^{-1}(K) \subseteq V_x$ for all $x \in A$, then $f^{-1}(K) \subseteq V$, and so $K \subseteq Y \backslash f(X \backslash V)$. Let $a_0 \in A$ be. Since $a_0 \notin V_{a_0}$ then $a_0 \in X \backslash V$. Hence $u = f(a_0) \in f(X \backslash V)$ and $u \notin Y \backslash f(X \backslash V)$. Now, given that $f^{-1}(K) \subseteq X \backslash U_x$, for every $x \in A$, we have that $f^{-1}(K) \subseteq \bigcap_{x\in A}(X\backslash U_x) = X\backslash U$, and so $K \cap [Y\backslash f(X\backslash U)] = \emptyset$. Consequently $[\{u\} \cup K] \cap [Y\backslash f(X\backslash U)] = \{u\}$, $[\{u\} \cup K] \cap [Y\backslash f(X\backslash V)] = K$. Moreover $[Y\backslash f(X\backslash U)] \cap [Y\backslash f(X\backslash V)] = Y\backslash f[X\backslash (U\cap V)] \subseteq f(U\cap V)$, because f is sobreyective. Given that $f(U\cap V) \in f(\mathcal{I})$, we have that $[Y\backslash f(X\backslash U)] \cap [Y\backslash f(X\backslash V)] \in f(\mathcal{I})$. □

Theorem 3.11 If $f : X \to Y$ is a sobreyective function, $(Y, \beta, \mathcal{J})$ is an ideal space and $(X, f^{-1}(\beta), \mathcal{I}_{f,\mathcal{J}})$ is $\mathcal{P}$-regular, then $(Y, \beta, \mathcal{J})$ is $\mathcal{P}$-regular. Here $I_{f,\mathcal{J}} = \{A \subseteq X :$ there is a $J \in \mathcal{J}$ with $A \subseteq f^{-1}(J)\}$.

Proof. Suppose that $G \subseteq Y$ is closed and $b = f(a) \in Y\backslash G$. Since $a \in X\backslash f^{-1}(G)$ and $X\backslash f^{-1}(G) \in f^{-1}(\beta)$, there is a $\{U, V\} \subseteq f^{-1}(\beta)$ such that $U\cap V \in \mathcal{I}_{f,\mathcal{J}}$, $[\{a\} \cup f^{-1}(G)] \cap U = \{a\}$ and $[\{a\} \cup f^{-1}(G)] \cap V = f^{-1}(G)$. Let $\{A, B\} \subseteq \beta$ be such that $U = f^{-1}(A)$ and $V = f^{-1}(B)$. Given that $f^{-1}(G) \subseteq X\backslash U = f^{-1}(X\backslash A)$, then $G \subseteq Y\backslash A$. Hence $b \in A$, $G \subseteq B$, $b \notin B$ and $G \cap A = \emptyset$. On the other hand, there exists $J \in \mathcal{J}$ such that $f^{-1}(A \cap B) = U \cap V \subseteq f^{-1}(J)$. Given that f is sobreyective we obtain that $A \cap B \subseteq J$, and so $A \cap B \in \mathcal{J}$. □

In the remainder of this section we review the product and sums of $\mathcal{P}$-regular spaces.

Theorem 3.12 If $(X, \tau, \mathcal{I})$ and $(Y, \beta, \mathcal{L})$ are $\mathcal{P}$-regular, then the space $(X \times Y, \tau \times \beta, \mathcal{I} \otimes \mathcal{L})$ is $\mathcal{P}$-regular.

Proof. If $F \subseteq X \times Y$ is closed and if $(a, b) \in (X \times Y)\backslash F$, then there are $U \in \tau$ and $V \in \beta$ such that $(a, b) \in U \times V \subseteq (X \times Y)\backslash F$. Since $(X, \tau, \mathcal{I})$ and $(Y, \beta, \mathcal{L})$ are $\mathcal{P}$-regular spaces, there are $\{U_1, U_2\} \subseteq \tau$ and $\{V_1, V_2\} \subseteq \beta$ with $[\{a\} \cup (X\backslash U)] \cap U_1 = \{a\}$, $[\{a\} \cup (X\backslash U)] \cap U_2 = X\backslash U$, $[\{b\} \cup (Y\backslash V)] \cap V_1 = \{b\}$, $[\{b\} \cup (Y\backslash V)] \cap V_2 = Y\backslash V$, $U_1 \cap U_2 \in \mathcal{I}$ and $V_1 \cap V_2 \in \mathcal{L}$. Thus $(a, b) \in U_1 \times V_1$, with $U_1 \times V_1 \in \tau \times \beta$, and $F \subseteq (X \times Y)\backslash (U \times V) = [(X\backslash U) \times Y] \cup [X \times (Y\backslash V)] \subseteq (U_2 \times Y) \cup (X \times V_2) \in \tau \times \beta$. On the other hand, $(a, b) \notin (U_2 \times Y) \cup (X \times V_2)$, $F \cap (U_1 \times V_1) \subseteq F \cap (U \times V) = \emptyset$ and $(U_1 \times V_1) \cap [(U_2 \times Y) \cup (X \times V_2)] = [(U_1 \cap U_2) \times V_1] \cup [U_1 \times (V_1 \cap V_2)] \in \mathcal{I} \otimes \mathcal{L}$. □

Theorem 3.13 If $\{(X_\alpha, \tau_\alpha, \mathcal{I}_\alpha) : \alpha \in \Lambda\}$ is a nonempty family of nonempty $\mathcal{P}$-regular spaces, then $\left(\prod_{\alpha\in\Lambda} X_\alpha, \prod_{\alpha\in\Lambda} \tau_\alpha, \bigotimes_{\alpha\in\Lambda} \mathcal{I}_\alpha\right)$ is $\mathcal{P}$-regular.

Proof. Let $X = \prod\limits_{\alpha\in\Lambda} X_\alpha$, $\tau = \prod\limits_{\alpha\in\Lambda} \tau_\alpha$ and $\mathcal{I} = \bigotimes\limits_{\alpha\in\Lambda} \mathcal{I}_\alpha$. Suppose that $F \subseteq X$ is closed and $a = (a_\alpha)_{\alpha\in\Lambda} \in X\backslash F$. There is a finite $\Lambda_0 \subseteq \Lambda$ such that, for every $\alpha \in \Lambda_0$, there exists $U_\alpha \in \tau_\alpha$ with $a \in \bigcap\limits_{\alpha\in\Lambda_0} p_\alpha^{-1}(U_\alpha) \subseteq X\backslash F$, where p_α is the α-th proyection. Since, for all $\alpha \in \Lambda_0$, $a_\alpha \notin X_\alpha\backslash U_\alpha$, there is a $\{V_\alpha, W_\alpha\} \subseteq \tau_\alpha$ such that $[\{a_\alpha\} \cup (X_\alpha\backslash U_\alpha)] \cap V_\alpha = \{a_\alpha\}$, $[\{a_\alpha\} \cup (X_\alpha\backslash U_\alpha)] \cap W_\alpha = X_\alpha\backslash U_\alpha$ and $V_\alpha \cap W_\alpha \in \mathcal{I}_\alpha$. It is observed that $V_\alpha \subseteq U_\alpha$ and $X_\alpha\backslash U_\alpha \subseteq W_\alpha$, for each $\alpha \in \Lambda_0$. If $V = \bigcap\limits_{\alpha\in\Lambda_0} p_\alpha^{-1}(V_\alpha)$ and $W = \bigcup\limits_{\alpha\in\Lambda_0} p_\alpha^{-1}(W_\alpha)$ then $a \in V$, $F \subseteq \bigcup\limits_{\alpha\in\Lambda_0} p_\alpha^{-1}(X_\alpha\backslash U_\alpha) \subseteq \bigcup\limits_{\alpha\in\Lambda_0} p_\alpha^{-1}(W_\alpha) = W$ and $F\cap V \subseteq \bigcup\limits_{\alpha\in\Lambda_0} [V \cap p_\alpha^{-1}(X_\alpha\backslash V_\alpha)] \subseteq \bigcup\limits_{\alpha\in\Lambda_0} p_\alpha^{-1}[V_\alpha \cap (X_\alpha\backslash V_\alpha)] = \emptyset$. Thus $F\cap V = \emptyset$. Now, suppose that $a \in W$. Then there is a $\delta \in \Lambda_0$ such that $a \in p_\delta^{-1}(W_\delta) \cap p_\delta^{-1}(V_\delta)$, and so $a_\delta \in [\{a_\delta\} \cup (X_\delta\backslash U_\delta)] \cap W_\delta = X_\delta\backslash U_\delta \subseteq X_\delta\backslash V_\delta$, but this is impossible. Hence $a \notin W$. Finally, $V\cap W \subseteq \bigcap\limits_{\alpha\in\Lambda_0} p_\alpha^{-1}(V_\alpha \cap W_\alpha)$, and so $V \cap W \in \mathcal{I}$. $\square$

Theorem 3.14 If $\{(X_i, \tau_i, \mathcal{I}_i) : i \in \Lambda\}$ is a nonempty collection of nonempty $\mathcal{P}$-regular spaces, with $X_i \cap X_j = \emptyset$ for each $i \neq j$, then the space $\left(\sum\limits_{i\in\Lambda} X_i, \sum\limits_{i\in\Lambda} \tau_i, \sum\limits_{i\in\Lambda} \mathcal{I}_i\right)$ is $\mathcal{P}$-regular.

Proof. Suppose that $F \subseteq \sum\limits_{i\in\Lambda} X_i$ is closed and that $a \in \left(\sum\limits_{i\in\Lambda} X_i\right)\backslash F$. Let $\alpha \in \Lambda$ be such that $a \in X_\alpha$. Since $F \cap X_\alpha$ is closed in X_α and $a \in X_\alpha\backslash(F \cap X_\alpha)$, there is a $\{U_\alpha, V_\alpha\} \subseteq \tau_\alpha \subseteq \sum\limits_{i\in\Lambda} \tau_i$ with $[\{a\} \cup (F \cap X_\alpha)] \cap U_\alpha = \{a\}$, $[\{a\} \cup (F \cap X_\alpha)] \cap V_\alpha = F \cap X_\alpha$ and $U_\alpha \cap V_\alpha \in \mathcal{I}_\alpha$. If we do $V = \sum\limits_{i\in\Lambda} W_i$, where $W_i = X_i$ if $i \neq \alpha$, and $W_\alpha = V_\alpha$, then $V \in \sum\limits_{i\in\Lambda} \tau_i$, $[\{a\} \cup F]\cap V = F$, $[\{a\} \cup F]\cap U_\alpha = \{a\}$ and $U_\alpha\cap V = U_\alpha\cap V_\alpha \in \mathcal{I}_\alpha \subseteq \sum\limits_{i\in\Lambda} \mathcal{I}_i$. $\square$

4. $\mathcal{P}$-normal ideal spaces

The $\mathcal{I}$-normal spaces were defined by Renuka Devi and Sivaraj in [8], while the $\mathcal{J}$-normal spaces were defined by Suriyakala and Vembu in [10]. In this section we introduce and study the $\mathcal{P}$-normal spaces, a weak form of the $\mathcal{J}$-normal spaces. The concepts of $\mathcal{P}$-normality and $\mathcal{I}$-normality turn out to be independent. An ideal space $(X, \tau, \mathcal{I})$ is said to be: $(i)\,\mathcal{I}$-normal if for every disjoint closed sets F and G, there are disjoint open sets U and V such that $\{F\backslash U, G\backslash V\} \subseteq \mathcal{I}$, and $(ii)\,\mathcal{J}$-normal if (X, τ) is T_1 and if, for every disjoint closed sets F and G, there is a $\{U, V\} \subseteq \tau$ such that $F \subseteq U$, $G \subseteq V$ and $U \cap V \in \mathcal{I}$. In the second case we can find a situation in which, for some disjoint closed sets F and G, $F \cup G \subseteq U$ or $F \cup G \subseteq V$, and this is not very desired. Some interesting additional properties of $\mathcal{J}$-normal spaces are presented in [7].

Definition 4.1 The ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{P}$-normal if for every disjoint closed sets F and G, there exists $\{U, V\} \subseteq \tau$ such that $F \subseteq U\backslash V$, $G \subseteq V\backslash U$ and $U \cap V \in \mathcal{I}$. A set $A \subseteq X$ is $\mathcal{P}$-normal if $(A, \tau_A, \mathcal{I}_A)$ is $\mathcal{P}$-normal.
It is observed that if $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-normal and $A \subseteq X$ is closed, then A is $\mathcal{P}$-normal. Also it is noted that Normal $\to$ $\mathcal{P}$-normal and that $(X, \tau, \{\emptyset\})$ is $\mathcal{P}$-normal if and only if (X, τ) is normal. Now, if $(X, \tau, \mathcal{I})$ is $\mathcal{J}$-normal then $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-normal, because if F and G are disjoint closed sets then there exists $\{U, V\} \subseteq \tau$ such that $F \subseteq U$, $G \subseteq V$ and $U \cap V \in \mathcal{I}$. Then $(F \cup G) \cap (U\backslash G) = F$, $(F \cup G) \cap (V\backslash F) = G$, and $(U\backslash G) \cap (V\backslash F) \in \mathcal{I}$. A little later we show that the reciprocal affirmation is, in general, false.
Then $T_4 \to \mathcal{J}$-normal $\to \mathcal{P}$-normal.

Example 4.2 1) (A$\mathcal{P}$-normal but not $\mathcal{I}$-normal or $\mathcal{J}$-normal space) If $X = \{a, b, c, d, e\}$, $\tau = \{\emptyset, \{c, e\}, \{c, d, e\}, \{a, b, c, e\}, X\}$ and $\mathcal{I} = \mathcal{P}(\{c, e\})$, then the only nonempty and disjoint closed sets are $\{a, b\}$ and $\{d\}$. If we do $U = \{a, b, c, e\}$ and $V = \{c, d, e\}$, we have that $(\{a, b\} \cup \{d\}) \cap V = \{d\}$, $(\{a, b\} \cup \{d\}) \cap U = \{a, b\}$ and $U \cap V = \{c, e\} \in \mathcal{I}$. Hence $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-normal.
Note that this space is not $\mathcal{P}$-regular because $c \notin \{d\}$, but there is no an open set W such that $W \cap \{c, d\} = \{d\}$. Moreover, this space is not $\mathcal{I}$-normal since there are not disjoint open sets U and V such that $\{a, b\}\backslash U \in \mathcal{I}$ and $\{d\}\backslash V \in \mathcal{I}$. In addition, this space is not $\mathcal{J}$-normal given that (X, τ) is not T_1.
2) If X is an infinite set, β is the cofinite topology in X, and if $\mathcal{I} = \mathcal{P}_f(X)$, then the space $(X, \beta, \mathcal{I})$ is not $\mathcal{P}$-normal.
3) (An $\mathcal{I}$-normal but not $\mathcal{P}$-normal space) In the set $\mathbf{Z}$ we consider the ideal $\mathcal{I} = \mathcal{P}(A)$, where $A = \{2n : n \in \mathbf{Z}\}$, and the topology β of all the sets $U \subseteq \mathbf{Z}$ such that, for each $n \in \mathbf{Z}$, if $2n \in U$ then $\{2n-1, 2n+1\} \subseteq U$. Then:
i) If F and G are disjoint closed sets and if $U = F \cap \{2n+1 : n \in \mathbf{Z}\}$ and

$V = G \cap \{2n+1 : n \in \mathbf{Z}\}$, we have that $\{U, V\} \subseteq \beta$, $\{F \backslash U, G \backslash V\} \subseteq \mathcal{I}$ and $U \cap V = \emptyset$. Hence $(\mathbf{Z}, \beta, \mathcal{I})$ is an $\mathcal{I}$-normal space.
ii) If $F = \{0, 1, 2\}$ and $G = \{4, 5, 6\}$ then F and G are closed sets. If $\{W, T\} \subseteq \beta$, with $F \subseteq W$ and $G \subseteq T$, then $3 \in W \cap T$ and so $W \cap T \notin \mathcal{I}$. Consecuently $(\mathbf{Z}, \beta, \mathcal{I})$ is not $\mathcal{P}$-normal.
4) (A $\mathcal{P}$-normal and not $\mathcal{J}$-normal space) If $X = \{a, b, c\}$ and $\lambda = \{\emptyset, X, \{a, b\}, \{c\}\}$, then $(X, \lambda, \{\emptyset\})$ is a $\mathcal{P}$-normal and not $\mathcal{J}$-normal space.

Note that $\mathcal{I}$-normal and $\mathcal{P}$-normal are independent concepts.
Next we present a characterization of $\mathcal{P}$-normality.

Theorem 4.3 The space $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-normal if and only if for each closed set $F \subseteq X$ and each $W \in \tau$, if $F \subseteq W$ then there are a $U \in \tau$ and a closed set $G \subseteq X$, such that $F \subseteq U \subseteq W$, $F \subseteq G \subseteq W$ and $U \backslash G \in \mathcal{I}$.

Proof. $(\rightarrow)$ If $F \subseteq X$ is closed and $F \subseteq W \in \tau$ then there exists $\{U, V\} \subseteq \tau$ such that $(F \cup (X \backslash W)) \cap U = F$, $(F \cup (X \backslash W)) \cap V = X \backslash W$ and $U \cap V \in \mathcal{I}$. This implies that $F \subseteq U \subseteq W$, $F \subseteq \acute{X} \backslash V \subseteq W$ and $U \backslash (X \backslash V) \in \mathcal{I}$.

$(\leftarrow)$ If F and G are disjoint closed sets then there are $U \in \tau$ and a closed set H such that $F \subseteq U \subseteq X \backslash G$, $F \subseteq H \subseteq X \backslash G$ and $U \backslash H \in \mathcal{I}$. Hence $(F \cup G) \cap U = F$, $(F \cup G) \cap (X \backslash H) = G$ and $U \cap (X \backslash H) \in \mathcal{I}$. □

Now we show some characterizations of $\mathcal{P}$-normality for closed ideals.

Theorem 4.4 If $\mathcal{I}$ is a closed ideal in (X, τ) then the following statements are equivalents:
1) The ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-normal.
2) For each closed set F and each $W \in \tau$, if $F \subseteq W$ then there are $U \in \tau$ and a closed set G such that $F \subseteq U \subseteq W$, $F \subseteq G \subseteq W$ and $\overline{U} \backslash G \in \mathcal{I}$.
3) For every disjoint closed sets F and G, there exists $\{U, V\} \subseteq \tau$ such that $(F \cup G) \cap U = F$, $(F \cup G) \cap V = G$ and $\overline{U} \cap \overline{V} \in \mathcal{I}$.

Proof.
1) $\rightarrow$ 3) Suppose that F and G are disjoint closed sets. There is a $\{U, V\} \subseteq \tau$ such that $(F \cup G) \cap U = F$, $(F \cup G) \cap V = G$ and $U \cap V \in \mathcal{I}$. Since $F \cap (X \backslash U) = \emptyset = G \cap (X \backslash V)$, there are $\{U_1, U_2, V_1, V_2\} \subseteq \tau$ and $\{I, J\} \subseteq \mathcal{I}$ such that $(F \cup (X \backslash U)) \cap U_1 = F$, $(F \cup (X \backslash U)) \cap U_2 = X \backslash U$, $(G \cup (X \backslash V)) \cap V_1 = G$, $(G \cup (X \backslash V)) \cap V_2 = X \backslash V$, $U_1 \cap U_2 = I$ and $V_1 \cap V_2 = J$. Therefore $F \subseteq U_1 \subseteq \overline{U_1} \subseteq (X \backslash U_2) \cup \overline{I} \subseteq U \cup \overline{I}$, $G \subseteq V_1 \subseteq \overline{V_1} \subseteq (X \backslash V_2) \cup \overline{J} \subseteq V \cup \overline{I}$ and

also $\overline{U_1} \cap \overline{V_1} \subseteq \left(U \cup \overline{I}\right) \cap \left(V \cup \overline{J}\right) = (U \cap V) \cup \left(U \cap \overline{J}\right) \cup \left(V \cap \overline{I}\right) \cup \left(\overline{I} \cap \overline{J}\right) \in \mathcal{I}$, because $\left\{\overline{I}, \overline{J}\right\} \subseteq \mathcal{I}$. Thus $\overline{U_1} \cap \overline{V_1} \in \mathcal{I}$. Moreover $G \cap U_1 \subseteq G \cap U = \emptyset$ and $F \cap V_1 \subseteq F \cap V = \emptyset$. 3) $\rightarrow$ 2) If $F \subseteq X$ is closed, $W \in \tau$ and $F \subseteq W$, there is a $\{U, V\} \subseteq \tau$ such that $(F \cup (X \backslash W)) \cap U = F$, $(F \cup (X \backslash W)) \cap V = X \backslash W$ and $\overline{U} \cap \overline{V} \in \mathcal{I}$. In this way, $F \subseteq U \subseteq W$, $F \subseteq X \backslash V \subseteq W$ and $\overline{U} \backslash (X \backslash V) = \overline{U} \cap V \subseteq \overline{U} \cap \overline{V} \in \mathcal{I}$. Hence $\overline{U} \backslash (X \backslash V) \in \mathcal{I}$. 2) $\rightarrow$ 1) This is immediate if we apply Theorem 4.3. □

Next we present a condition under which $\mathcal{P}$-regular implies $\mathcal{P}$-normal.

Theorem 4.5 If the space $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-regular and if (X, τ) is compact, then $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-normal.

Proof. Suppose that F and G are disjoint closed sets. For each $f \in F$ there exists $\{U_f, V_f\} \subseteq \tau$ such that $(\{f\} \cup G) \cap U_f = \{f\}$, $(\{f\} \cup G) \cap V_f = G$ and $U_f \cap V_f \in \mathcal{I}$. Given that F is compact, there is a finite $F_0 \subseteq F$ such that $F \subseteq U = \bigcup\limits_{f \in F_0} U_f$. If $V = \bigcap\limits_{f \in F_0} V_f$ then $G \subseteq V$, $U \cap G = \emptyset$ and $U \cap V \in \mathcal{I}$. Similarly, there is a $\{U_1, V_1\} \subseteq \tau$ such that $F \subseteq U_1$, $G \subseteq V_1$, $V_1 \cap F = \emptyset$ and $U_1 \cap V_1 \in \mathcal{I}$. In this way $(F \cup G) \cap (U \cap U_1) = F$, $(F \cup G) \cap (V \cap V_1) = G$ and $(U \cap U_1) \cap (V \cap V_1) = (U \cap V) \cap (U_1 \cap V_1) \in \mathcal{I}$. □

Now we show some functional properties of $\mathcal{P}$-normality.

Theorem 4.6
1) If $f : (X, \tau) \rightarrow (Y, \beta)$ is continuous, sobreyective and closed, and if $(X, \tau, \mathcal{I})$ is $\mathcal{P}$-normal, then $(Y, \beta, f(\mathcal{I}))$ is $\mathcal{P}$-normal.
2) If $f : X \rightarrow Y$ is sobreyective, $(Y, \beta, \mathcal{L})$ is an ideal space and if $\left(X, f^{-1}(\beta), \mathcal{I}_{f,\mathcal{L}}\right)$ is $\mathcal{P}$-normal, then $(Y, \beta, \mathcal{L})$ is $\mathcal{P}$-normal.

Proof.
1) Suppose that L and K are disjoint closed subsets of Y. There is a $\{U, V\} \subseteq \tau$ such that $\left[f^{-1}(L) \cup f^{-1}(K)\right] \cap U = f^{-1}(L)$, $\left[f^{-1}(L) \cup f^{-1}(K)\right] \cap V = f^{-1}(K)$ and $U \cap V \in \mathcal{I}$. Hence $L \subseteq Y \backslash f(X \backslash U)$, $K \subseteq Y \backslash f(X \backslash V)$, $L \subseteq f(X \backslash V)$ and $K \subseteq f(X \backslash U)$, given that f is obreyective. Thus $L \cap [Y \backslash f(X \backslash V)] = \emptyset$ and $K \cap [Y \backslash f(X \backslash U)] = \emptyset$. Now $[Y \backslash f(X \backslash U)] \cap [Y \backslash f(X \backslash V)]$
$= Y \backslash f(X \backslash (U \cap V))$. Since f is sobreyective, $Y \backslash f(X \backslash (U \cap V)) \subseteq f(U \cap V)$, with $f(U \cap V) \in f(\mathcal{I})$. Hence $[Y \backslash f(X \backslash U)] \cap [Y \backslash f(X \backslash V)] \in f(\mathcal{I})$.
2) This is immediate. □

We will finish by reviewing the sum of $\mathcal{P}$-normal spaces.

Theorem 4.7 If $\{(X_i, \tau_i, \mathcal{I}_i) : i \in \Lambda\}$ is a nonempty collection of nonempty $\mathcal{P}$-normal spaces, with $X_i \cap X_j = \emptyset$ for each $i \neq j$, then the space $\left(\sum\limits_{i\in\Lambda} X_i, \sum\limits_{i\in\Lambda} \tau_i, \sum\limits_{i\in\Lambda} \mathcal{I}_i\right)$ is $\mathcal{P}$-normal.

Proof. Let $X = \sum\limits_{i\in\Lambda} X_i$, $\tau = \sum\limits_{i\in\Lambda} \tau_i$ and $\mathcal{I} = \sum\limits_{i\in\Lambda} \mathcal{I}_i$. If F and G are disjoint closed sets in (X, τ), then $F \cap X_i$ and $G \cap X_i$ are disjoint closed sets in (X_i, τ_i), for each $i \in \Lambda$. Hence, for each $i \in \Lambda$, there is a $\{U_i, V_i\} \subseteq \tau_i$ such that $[(F \cap X_i) \cup (G \cap X_i)] \cap U_i = F \cap X_i$, $[(F \cap X_i) \cup (G \cap X_i)] \cap V_i = G \cap X_i$ and $U_i \cap V_i \in \mathcal{I}_i$. If we do $U = \sum\limits_{i\in\Lambda} U_i$ and $V = \sum\limits_{i\in\Lambda} V_i$ then $(F \cup G) \cap U = \sum\limits_{i\in\Lambda} (F \cap U_i) = F$, $(F \cup G) \cap V = \sum\limits_{i\in\Lambda} (G \cap V_i) = G$ and $U \cap V = \sum\limits_{i\in\Lambda} (U_i \cap V_i) \in \mathcal{I}$.

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