International Journal of Pure and Applied Mathematics
Volume 107 No. 4 2016, 909-925
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v107i4.9

ijpam.eu

ON EXPONENTIAL CONVERGENCE OF GENERIC QUANTUM MARKOV SEMIGROUPS IN A WASSERSTEIN-TYPE DISTANCE

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Abstract: We investigate about exponential convergence for generic quantum Markov semigroups using an generalization of the Lipschitz seminorm and a noncommutative analogue of Wasserstein distance. We show turns out to be closely related with classical convergence rate of reductions to diagonal subalgebras of the given generic quantum Markov semigroups.In particular we compute the convergence rates of generic quantum Markov semigroups.

AMS Subject Classification: 81S22, 60J27
Key Words: quantum Markov semigroups, Wasserstein distance, exponential convergence

1. Introduction

We consider the von Neumann algebra $\mathbf{B}(\mathsf{h})$ of all linear bounded operators on a given complex separable Hilbert space h and a Quantum Markov semigroup (QMS) $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ which acts on $\mathbf{B}(\mathsf{h})$, i.e., $\mathcal{T}$ is a weakly*-continuous

Received: January 30, 2016
Published: May 5, 2016

semigroup of completely positive, preserving, normal maps on $\mathbf{B}(\mathsf{h})$. Quantum Markov semigroups (QMS) are a non-commutative extension of Markov semigroups defined in classical probability, they represent an evolution without memory of a microscopic system in accordance with the laws of quantum physics and fit into the framework of open quantum systems. The semigroup $\mathcal{T}$ corresponds to the Heisenberg picture in the sense that given any observable x, $\mathcal{T}_t(x)$ describes its evolution at time t. In this way, given a density matrix ρ, its dynamics (Schrodinger picture) is given by the semigroup $\mathcal{T}_{*t}(\rho)$, where $tr(\rho\mathcal{T}_t(x)) = tr(\mathcal{T}_{*t}(\rho)x)$.

Several aspects of temporal evolutions described by QMSs have been investigated. By example, in [4],[5], and [6], the exponential speed of convergence of the quantum Markov semigroup is studied using the quantum L^2-spectral gap ($gap(\mathcal{L})$). In [3] a Wasserstein-type distance, denoted by W_d, has been defined and applied to measure deviations from equilibrium, in other words, to define an entropy production index (see [16, 17]). W_d is a non commutative analogue of the classical Wasserstein distance w_d used in optimal transport (see [12],[22], [23]).

In this paper we use a generalization of the Lipschitz seminorm and a non-commutative analogue of Wasserstein distance to study exponential convergence of generic QMSs. This research is motivated by the exploration of relation between exponential convergence of QMSs and his classical reductions given by classical Markov semigroups. The exponential convergence in the classical case is represented by a Wasserstein curvature (or Chen exponent) σ_d linked with the classical Wasserstein distance (see [8],[9],[19] ,[22]). Moreover, we show that in the generic QMSs case the exponent convergence is related with σ_d and the parameters of QMS.

The paper is organized as follows. In Section 2 we recall the basic aspects about classical Wasserstein distance. We recall generalization of the Lipschitz seminorm and a noncommutative analogue of Wasserstein distance introduced in [3]. After, some useful estimates on norms of commutators are showed in the Section 4. Finally, we apply these estimates. Specifically, we see in Section 5 that if $\mathcal{T}$ is a generic quantum Markov semigroup and

$$r = \sup_{n \neq m} \max \left\{ \frac{d(m, m+1)}{d(n, m)}, \frac{d(m, m-1)}{d(n, m)} \right\} < \infty$$

then for all states ρ_1, ρ_2

$$W_d(\mathcal{T}_{*t}(\rho_1), \mathcal{T}_{*t}(\rho_2)) \leq (4\sqrt{2} + 1 + 2\sqrt{2}\, r)e^{-tk} W_d(\rho_1, \rho_2)$$

with $k := \min_{n\neq m}\left\{\frac{\mu_n+\lambda_n+\mu_m+\lambda_m}{2}\wedge\sigma_d\right\}$. Where μ_n, λ_n are coefficients generator of semigroup and σ_d is a rate convergence of classical reduction of semigroup.

2. Exponential Convergence: Classical Case

We start this section by reviewing the Wasserstein distance and Wasserstein curvature for classical Markov processes.

Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$ be a filtered probability space, E a Polish space endowed with metric d, and $\mathcal{A} = L^\infty(E)$. Consider a E-valued cadlag Markov process $\{(X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E}\}$, with $(T_t)_{t\geq 0}$ associated Markov semigroup acting on $\mathcal{A}$ as follows

$$T_t f(x) = \int_E f(y)P_t(x, dy), \quad P_t(x, dy) = \mathbb{P}_x(X_t \in dy), \quad x \in E.$$

The predual semigroup of $(T_t)_{t\geq 0}$ acts on probability measures μ as

$$T_{*t}\mu(\cdot) = \int_E \mu(dx)P_t(x, \cdot).$$

We denote by $\mathcal{P}_d(E)$ the space of probability measures ν on E such that

$$\int_E d(x, y)\nu(dy) < +\infty \text{ for some (or equivalently for all) } x \in E.$$

Moreover, we consider $\mathbf{Lip}_d(E)$ the space of Lipschitz functions on E with a Lipschitz seminorm defined by

$$\|f\|_{Lip_d} := \sup_{x\neq y}\frac{|f(x) - f(y)|}{d(x, y)} < +\infty.$$

Remark 1. Under the previous assumptions, if a Markov kernel $P_t(x, \cdot)$ belongs to $\mathcal{P}_d(E)$ for all $t > 0$ and for all $x \in E$ then $T_t(f)$ is well defined for all $f \in \mathbf{Lip}_d(E)$.

We can therefore define

$$\overline{\sigma}_d(t) := -\sup\{\log\|T_t f\|_{Lip_d}; \|f\|_{Lip_d} = 1\}, \quad t \geq 0.$$

Remark 2. Is easy to see that

$$\overline{\sigma}_d(t) = -\sup\{\log\|T_t f\|_{Lip_d}; \|f\|_{Lip_d} = 1, \overline{f} = f\}, \quad t \geq 0.$$

i.e, the supremum does not change if we restrict to self-adjoint elements.

Note that $\overline{\sigma}_d(0) = 0$. By the semigroup property of T_t, it follows that the function $\overline{\sigma}_d(t)$ is super-additive so that the following limit is well defined:

$$\sigma_d := \lim_{t \downarrow 0} \frac{\overline{\sigma}_d(t)}{t} = \inf_{t>0} \frac{\overline{\sigma}_d(t)}{t}. \tag{1}$$

Moreover, the number σ_d is the best (maximal) constant Δ in the contraction inequality

$$\|T_t f\|_{Lip_d} \leq e^{-\Delta t} \|f\|_{Lip_d}, \quad f \in \mathbf{Lip}_d(E), \quad t > 0. \tag{2}$$

Definition 1. The number σ_d given by (1) is called *Wasserstein curvature* of the process $(X_t)_{t \geq 0}$ with respect to metric d.

This notion of curvature was introduced by Joulin [19],[20] and Ollivier [21] and is connected to the notion of Ricci curvature on Riemannian manifolds [24]. In this remainder of this section, we will assume implicitly that the Markov kernel $P_t(x, \cdot)$ belongs to the space $\mathcal{P}_d(E)$ for all $t > 0, x \in E$.

The coefficient σ_d is linked with the classical Wasserstein distance.

Remark 3. The classical Wasserstein distance is defined by

$$w_d(\mu, \nu) = \inf_{\vartheta \in \Xi(\mu,\nu)} \int_{M \times M} d(m,n) d\vartheta(m,n)$$

where (M, d) is a metric space and $\Xi(\mu, \nu)$ is the set of all Borel probability measures ϑ on $M \times M$ such that for all measurable subsets $A, B \subseteq M$

$$\vartheta(A \times M) = \mu(A), \quad \vartheta(M \times B) = \nu(B).$$

When M is a separable space and $\mu, \nu \in \mathcal{P}_d(M)$ the Kantorovich-Rubinstein theorem provides another representation for the Wasserstein metric:

$$w_d(\mu, \nu) = \sup \left\{ \int_M f d(\mu - \nu); \quad f \in L_1(d|\mu - \nu|); \quad \| f \|_{Lip_d} \leq 1 \right\}$$

(for a proof of the Kantorovich-Rubinstein theorem see for example [12], Theorem 11.8.2 p.421).

By remark 3, the Wasserstein curvature σ_d is the best (maximal) constant Δ in the inequality

$$w_d(T_{*t}(\mu_1), T_{*t}(\mu_2)) \leq e^{-\Delta t} d(\mu_1, \mu_2), \quad t > 0, \tag{3}$$

where μ_1 and μ_2 are σ-finite measures.

Then σ_d is the best (maximal) constant Δ holding simultaneously the inequalities (2) and (3).

3. An Non Commutative Extension of the Lipschitz Seminorm and a Wasserstein-Type Distance

We start our discussion about a non commutative extension of the Lipschitz seminorm and a Wasserstein-type distance. In the quantum case, we consider h complex separable Hilbert space with orthonormal basis fixed $(e_k)_{k\in V}$ (V is a finite or countable set).

In [3], we proposed a quantum version of w_d, we recall the definition.

Definition 2. The *quantum Wasserstein distance* between two states

$$\varphi_{\sigma_1}(\cdot) = tr(\sigma_1(\cdot)), \quad \varphi_{\sigma_2}(\cdot) = tr(\sigma_2(\cdot)) \quad \text{in } \mathsf{B}(\mathsf{h})$$

is defined by:

$$W_d\left(\sigma_1, \sigma_2\right) := W_d\left(\varphi_{\sigma_1}, \varphi_{\sigma_2}\right) = \sup_{\|a\|_{LIP_d}\leq 1} \left|tr\left(\left(\sigma_1 - \sigma_2\right) a\right)\right|$$

with

$$\|a\|_{LIP_d} = \sup_{m,l\in V, m\neq l} \| \, \delta^d_{ml}(a) \, \|, \quad \delta^d_{ml}(a) = \frac{1}{d(m,l)} \left[(|e_m\rangle\langle e_l| + |e_l\rangle\langle e_m|), a\right],$$

and d a distance defined on the set V.

Note that the usual deriviation $\delta_{ml}(a) = [(|e_m\rangle\langle e_l| + |e_l\rangle\langle e_m|), a]$ satisfies $\delta_{ml} = d(m,l)\delta^d_{ml}$.

We collect here some preliminary results on the quantum Wasserstein distance that we need in the sequel.

Proposition 3. *The quantum Wasserstein distance $W_d(\varphi_{\sigma_1}, \varphi_{\sigma_2})$ is equal to the infimum of $|tr\left((\sigma_1 - \sigma_2)a\right)|$ on* self-adjoints *elements $a \in \mathsf{B}(\mathsf{h})$ with $\|a\|_{LIP_d} \leq 1$*

Proof. To prove our statement is enough to suppose that $W_d(\varphi_{\sigma_1}, \varphi_{\sigma_2}) < \infty$ (the procedure is analogous if $W_d(\varphi_{\sigma_1}, \varphi_{\sigma_2}) = \infty$). For any $\epsilon > 0$ there exists $a \in \mathsf{B}(\mathsf{h})$ with $\|a\|_{LIP_d} \leq 1$

$$W_d(\varphi_{\sigma_1}, \varphi_{\sigma_2}) - \epsilon < |tr\left((\sigma_1 - \sigma_2)a\right)| = |tr\left((\sigma_1 - \sigma_2)a^*\right)|.$$

Let θ be the phase of the complex number $tr\left((\sigma_1 - \sigma_2)a\right)$ so that

$$e^{-i\theta} tr((\sigma_1 - \sigma_2)a) = |tr((\sigma_1 - \sigma_2)a)| = e^{i\theta} tr((\sigma_1 - \sigma_2)a^*), \tag{4}$$

The operator $y = (e^{-i\theta}a + e^{i\theta}a^*)/2$ is clearly self-adjoint and has Lipschitz norm smaller than 1, indeed

$$\sup_{m,l} \|\delta^d_{ml}(y)\| \leq \frac{1}{2} \sup_{m,l} \Big(\|\delta^d_{ml}(a)\| + \|\delta^d_{ml}(a^*)\| \Big) \leq 1.$$

Moreover, by (4),

$$|tr\,((\sigma_1 - \sigma_2)y)\,| = |tr\,((\sigma_1 - \sigma_2)a)\,| > W(\varphi_{\sigma_1}, \varphi_{\sigma_2}) - \epsilon.$$

This completes the proof. □

We call *diagonal algebra*, and denote it by $\mathcal{D}$ the Abelian algebra generated by one-dimensional projections $|e_k\rangle\langle e_k|$. Let $\mathcal{E} : \mathbf{B}(\mathbf{h}) \to \mathcal{D}$ be the conditional expectation with range $\mathcal{D}$ defined by

$$\mathcal{E}(x) = \sum_j x_{jj} |e_j\rangle\langle e_j| := \sum_j x(j) |e_j\rangle\langle e_j| \tag{5}$$

and let $\mathcal{E}_*$ be the predual map on trace class operators with range $l_1(V)$

$$\mathcal{E}_*(\omega) = \sum_j \omega_{jj} |e_j\rangle\langle e_j|. \tag{6}$$

Proposition 4. *For all $x \in \mathbf{B}(\mathbf{h})$ it follows that*

(a) $\|\mathcal{E}(x)\|_{LIP_d} = \sup_{m,l \in V, m \neq l} \frac{1}{d(m,l)} |x(l) - x(m)|$.

(b) $\|\mathcal{E}(x)\|_{LIP_d} \leq \|x\|_{LIP_d}$

Proof. (a) If $x \in \mathbf{B}(\mathbf{h})$ then $\mathcal{E}(x) = \sum_{s \in V} x(s) |e_s\rangle\langle e_s|$ where $x(s) \in \mathbb{C}$ and the convergence of the sum is in the weak* topology, then

$$d(n,m)\delta^d_{mn}(\mathcal{E}(x)) = (x(l) - x(m))|e_l\rangle\langle e_m| - (x(l) - x(m))|e_m\rangle\langle e_l|.$$

Since the norm of an anti self-adjoint matrix is the largest eigenvalue, computing the norm of the above 2×2 matrix (thought as an operator on the linear span of e_l, e_m) we find

$$\begin{aligned}
\|\mathcal{E}(x)\|_{LIP_d} &= \sup_{m,l \in V, m \neq l} \|\delta^d_{ml}(\mathcal{E}(x))\| \\
&= \sup_{m,l \in V, m \neq l} \frac{1}{d(m,l)} \|(x(l) - x(m))|e_l\rangle\langle e_m| - (x(l) - x(m))|e_m\rangle\langle e_l|\| \\
&= \sup_{m,l \in V, m \neq l} \frac{1}{d(m,l)} \sqrt{\||x(l) - x(m)|^2 |e_m\rangle\langle e_m| + |x(l) - x(m)|^2 |e_l\rangle\langle e_l|\|} \\
&= \sup_{m,l \in V, m \neq l} \frac{1}{d(m,l)} |x(l) - x(m)| = \|x(\cdot)\|_{LIP_d}.
\end{aligned}$$

(b) First notice that if $n \neq m$ and $x = \sum_{ij} x_{ij} |e_i\rangle\langle e_j|$ then

$$\begin{aligned} d(n,m)\delta^d_{mn}(x) &= (x_{nm} - x_{mn})|e_m\rangle\langle e_m| + (x_{mn} - x_{nm})|e_n\rangle\langle e_n| \\ &+ (x_{nn} - x_{mm})|e_m\rangle\langle e_n| + \sum_{\substack{j\neq n \\ j\neq m}} x_{nj}|e_m\rangle\langle e_j| \\ &+ (x_{mm} - x_{nn})|e_n\rangle\langle e_m| + \sum_{\substack{j\neq n \\ j\neq m}} x_{mj}|e_n\rangle\langle e_j| \\ &- \sum_{\substack{i\neq n \\ i\neq m}} x_{im}|e_i\rangle\langle e_n| - \sum_{\substack{i\neq m \\ i\neq n}} x_{in}|e_i\rangle\langle e_m|. \end{aligned}$$

Let p_{nm} be a projection defined by $p_{nm} = |e_n\rangle\langle e_n| + |e_m\rangle\langle e_m|$ so

$$\begin{aligned} d(n,m)p_{nm}\delta^d_{mn}(x)p_{nm} &= (x_{nm} - x_{mn})|e_m\rangle\langle e_m| + (x_{mn} - x_{nm})|e_n\rangle\langle e_n| \\ &+ (x_{nn} - x_{mm})|e_m\rangle\langle e_n| + (x_{mm} - x_{nn})|e_n\rangle\langle e_m| \\ &. \end{aligned}$$

Since the norm of an anti self-adjoint matrix is the largest eigenvalue, computing the norm of the above 2×2 matrix (thought of as an operator on the linear span of e_n, e_m) we find

$$\begin{aligned} d(n,m)^2 \|p_{nm}\delta^d_{mn}(x)p_{nm}\|^2 &= \|(|x_{nm} - x_{mn}|^2 + |x_{nn} - x_{mm}|^2) e_m\rangle\langle e_m| \\ &+ (2\Re e((\overline{x_{nm} - x_{mn}})(x_{nn} - x_{mm})))|e_m\rangle\langle e_n| \\ &+ (2\Re e((\overline{x_{nm} - x_{mn}})(x_{nn} - x_{mm})))|e_n\rangle\langle e_m| \\ &+ (|x_{nm} - x_{mn}|^2 + |x_{nn} - x_{mm}|^2)|e_n\rangle\langle e_n|\| \\ &= |x_{nm} - x_{mn}|^2 + |x_{nn} - x_{mm}|^2 \\ &+ \sqrt{(2\Re e((\overline{x_{nm} - x_{mn}})(x_{nn} - x_{mm})))^2}, \end{aligned}$$

so, by item (a), we see that $\|\delta^d_{mn}(\mathcal{E}(x))\| \leq \|p_{nm}\delta^d_{mn}(x)p_{nm}\|$,then

$$\|\delta^d_{mn}(\mathcal{E}(x))\| \leq \|p_{nm}\delta^d_{mn}(x)p_{nm}\| \leq \|p_{nm}\|\|\delta^d_{mn}(x)\|\|p_{nm}\| \leq \|\delta^d_{mn}(x)\|$$

therefore $\|\delta^d_{mn}(\mathcal{E}(x))\| \leq \|\delta^d_{mn}(x)\| \leq \|x\|_{LIP_d}$ for all n, m, $n \neq m$, so $\|\mathcal{E}(x))\|_{LIP_d} \leq \|x\|_{LIP_d}$.

□

Let σ_1, σ_2 be states on $\mathsf{B}(\mathsf{h})$ then $\mathcal{E}_*(\sigma_1), \mathcal{E}_*(\sigma_2)$ are diagonal states with respect to $(e_j)_{j\in V}$ (i.e. measures probabilities on V), then

$$tr((\sigma_1 - \sigma_2)\mathcal{E}(x)) = tr(\mathcal{E}_*(\sigma_1 - \sigma_2)\mathcal{E}(x)) = \sum_s x(s)(\sigma_1 - \sigma_2)(s)$$
$$= \int_V x(\cdot)d(\sigma_1(\cdot) - \sigma_2(\cdot)).$$

Note that, since $tr(\sigma_j|_{\mathcal{D}}\, a) = tr(\sigma_j|_{\mathcal{D}}\, \mathcal{E}(a))) = tr(\sigma_j \mathcal{E}(a))$ for all $a \in \mathsf{B}(\mathsf{h})$),

$$\begin{aligned} W_d(\sigma_1|_{\mathcal{D}}, \sigma_2|_{\mathcal{D}}) &= \sup_{a\in\mathcal{D}, a=a^*} |tr((\sigma_1 - \sigma_2)a)| \\ &= \sup_{a\in\mathcal{D}, a=a^*} |tr((\sigma_1 - \sigma_2)\mathcal{E}(a))| \\ &= w_d(\sigma_1(\cdot), \sigma_2(\cdot)) \end{aligned}$$

for all pairs (σ_1, σ_2) of states. Then the restriction of W_d to the diagonal subalgebra of $\mathsf{B}(\mathsf{h})$ coincides with the classical Wasserstein distance w_d.

4. Estimates of Lipschitz Seminorm

In this section we prove some useful estimates on the norms of commutators $\delta_{mn}(x)$. These estimates turn out to be useful for computing the exponential convergence rate of a generic quantum Markov semigroup. We begin by some simple lemma.

Lemma 5. *Let e, f be two unit vectors in h and $a, b \in \mathsf{h}$. Then*

$$\| \, |a\rangle\langle e| + |b\rangle\langle f| \, \|^2 = \frac{1}{2}\left(\|a\|^2 + \|b\|^2 + \left(\left(\|a\|^2 - \|b\|^2 \right)^2 + 4\, |\langle a, b\rangle|^2 \right)^{1/2} \right)$$

In particular

$$\frac{1}{2}\left(\|a\|^2 + \|b\|^2 \right) \leq \| \, |a\rangle\langle e| + |b\rangle\langle f| \, \|^2 \leq \|a\|^2 + \|b\|^2$$

Proof. Let $x = |a\rangle\langle e| + |b\rangle\langle f|$. Then x^*x is a rank-two self-adjoint operator that can be represented by the 2×2 matrix

$$\begin{pmatrix} \|a\|^2 & |\langle a, b\rangle| \\ |\langle a, b\rangle| & \|b\|^2 \end{pmatrix}$$

Recalling that $\|x\|^2 = \|x^*x\|$ and computing the biggest eigenvalue we obtain the squared norm of x. The last inequalities immediately follow from the Schwarz inequality $|\langle a, b\rangle| \leq \|a\| \cdot \|b\|$. □

The previous Lemma will be used to deduce bounds of $\|\delta_{nm}(x)\|$.

Proposition 6. *For all $n \neq m$ and all $x \in \mathsf{B}(\mathsf{h})$ we have*

$$\begin{aligned} \|\delta_{nm}(x)\|^2 &\leq 2\Bigg(2\,|x_{nn}-x_{mm}|^2 + 2\,|x_{nm}-x_{mn}|^2 \\ &+ \sum_{j\neq m,n} |x_{jn}|^2 + \sum_{j\neq m,n} |x_{jm}|^2 + \sum_{i\neq m,n} |x_{ni}|^2 + \sum_{i\neq m,n} |x_{mi}|^2 \Bigg) \end{aligned} \tag{7}$$

Proof. Note that the above series converge because x is a bounded operator. Computing

$$\delta_{mn}(x) = |e_m\rangle\langle x_{n\,\bullet}| + |e_m\rangle\langle x_{m\,\bullet}| - |x_{\bullet\,m}\rangle\langle e_n| - |x_{\bullet\,n}\rangle\langle e_m|$$

where $x_{n\,\bullet}$ and $x_{m\,\bullet}$ (resp. $x_{\bullet\,n}$ and $x_{\bullet\,m}$) denote the n and m row (resp. column) vector of x. Keeping into account cancellations for $i, j = n, m$ we find then

$$\delta_{mn}(x) = |e_m\rangle\langle\phi_n| + |e_n\rangle\langle\psi_m| - |\xi_m\rangle\langle e_n| - |\eta_n\rangle\langle e_m| \tag{8}$$

where

$$\begin{aligned} \phi_n &= (x_{n1}, \ldots, x_{n\,m-1}, x_{nm}-x_{mn}, x_{n\,m+1}, \ldots, x_{nn}-x_{mm}, x_{n\,n+1}, \ldots) \\ \psi_m &= (x_{m1}, \ldots, x_{m\,m-1}, x_{mm}-x_{nn}, x_{m\,m+1}, \ldots, x_{mn}-x_{nm}, x_{m\,n+1}, \ldots) \\ \xi_m &= (-x_{1m}, \ldots, -x_{m-1\,m}, 0, -x_{m+1\,m}, \ldots, -x_{n-1\,m}, 0, -x_{n+1\,m}, \ldots) \\ \eta_n &= (-x_{1n}, \ldots, -x_{m-1\,n}, 0, -x_{m+1\,n}, \ldots, -x_{n-1\,n}, 0, -x_{n+1\,n}, \ldots) \end{aligned}$$

It follows then, from Lemma 5 and the elementary inequality $\|x+y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$

$$\begin{aligned} \|\delta_{mn}(x)\|^2 &\leq 2\,\||e_m\rangle\langle\phi_n| + |e_n\rangle\langle\psi_m|\|^2 + 2\,\|(|\xi_m\rangle\langle e_n| + |\eta_n\rangle\langle e_m|)^*\|^2 \\ &\leq 2\left(\|\phi_n\|^2 + \|\psi_m\|^2 + \|\xi_m\|^2 + \|\eta_n\|^2\right). \end{aligned}$$

The proof is completed writing explicitly the norms of the four vectors $\phi_n, \psi_m, \xi_n, \eta_m$. □

Proposition 7. *For all $n \neq m$ and all $x \in \mathsf{B}(\mathsf{h})$ we have*

$$\begin{aligned} \|\delta_{nm}(x)\|^2 &\geq |x_{nn}-x_{mm}|^2 + |x_{nm}-x_{mn}|^2 \\ &+ \max\left\{ \sum_{i\neq m,n} |x_{ni}|^2, \sum_{i\neq m,n} |x_{mi}|^2 \right\} \end{aligned} \tag{9}$$

Proof. Let p_{mn} be the orthogonal projection onto the subspace generated by e_n and e_m Clearly, for all unit vector $u \in \mathsf{h}$ we have

$$\|\delta_{nm}(x)\|^2 \geq \|\delta_{nm}(x)u\|^2 \geq \|p_{nm}\delta_{nm}(x)u\|^2 .$$

Note that vectors ξ_n, η_m in (8) are orthogonal to e_n, e_m and so the right-hand side is equal to

$$\|\langle \phi_n, u\rangle\, e_n + \langle \psi_m, u\rangle\, e_m\|^2 = |\langle \phi_n, u\rangle|^2 + |\langle \psi_m, u\rangle|^2 .$$

Maximizing the right-hand side on the unit sphere in h we find

$$\|\delta_{nm}(x)\|^2 \geq \max\left\{ \|\phi_n\|^2 , \|\psi_m\|^2\right\}$$

and the claimed inequality follows computing the norms of ϕ_n and ψ_m □

For a self-adjoint x we can also find an upper bound for the norm of $\delta_{nm}(x)$ as a multiple of the right hand side of (9).

Theorem 8. *For all self-adjoint $x \in \mathcal{B}(\mathsf{h})$ and all n, m we have*

$$M_{nm}(x) \leq \|\delta_{nm}(x)\|^2 \leq 8\, M_{nm}(x)$$

where

$$M_{nm}(x) = |x_{nn} - x_{mm}|^2 + |x_{nm} - x_{mn}|^2 + \max\left\{ \sum_{i\neq m,n} |x_{ni}|^2, \sum_{i\neq m,n} |x_{mi}|^2 \right\}$$

Proof. It suffices to apply Propositions 6 and 7 noting that, for a self-adjoint operator x

$$\sum_{j\neq m,n} |x_{jn}|^2 + \sum_{j\neq m,n} |x_{jm}|^2 + \sum_{i\neq m,n} |x_{ni}|^2 + \sum_{i\neq m,n} |x_{mi}|^2$$

is dominated by

$$4 \max\left\{ \sum_{i\neq m,n} |x_{ni}|^2, \sum_{i\neq m,n} |x_{mi}|^2 \right\} .$$

□

Remark 4. A straightforward application of Theorem 8 shows that our Wasserstein norm is equivalent to the Hilbert-Schmidt norm for a finite V (with $\mathrm{card}(V) \geq 2$) and, more generally, for a set V with a distance d such that

$$\inf_{m,l\in V, m\neq l} d(m,l) > 0 \qquad \text{and} \qquad \sup_{m,l\in V} d(m,l) < \infty.$$

5. Lipschitz Seminorm and Generic QMSs

Generic QMS arise in the stochastic limit of a open discrete quantum system with generic Hamiltonian, interacting with Gaussian fields through a dipole type interaction (see Refs.[1],[2] and [6]).

The generator is given by

$$\mathcal{L}(x) = G^* x + \sum_{k,j;k\neq j} L_{kj}^* x L_{kj} + xG,$$

where

$$H = \sum_{k\in V} \kappa_k |e_k\rangle\langle e_k|,$$

$$\mu_k := \sum_{j\in V,\kappa_j<\kappa_k} \gamma_{kj}^-, \quad \lambda_k := \sum_{j\in V,\kappa_j<\kappa_k} \gamma_{jk}^+,$$

and operators G, L_{kj} given by

$$L_{kj} = \begin{cases} \sqrt{\gamma_{kj}^-}|e_j\rangle\langle e_k|, & \text{if } \kappa_j < \kappa_k \\ \sqrt{\gamma_{kj}^+}|e_j\rangle\langle e_k|, & \text{if } \kappa_k < \kappa_j, \end{cases}$$

$$G = -\sum_{k\in V}\left(\frac{\mu_k+\lambda_k}{2} + i\kappa_k\right)|e_k\rangle\langle e_k| = -\frac{1}{2}\sum_{k,j;k\neq j} L_{kj}^* L_{kj} - iH, \tag{10}$$

We denote by $\mathcal{D}$, and call it the *diagonal subalgebra*, the Abelian subalgebra of $\mathcal{B}(\mathsf{h})$ of operators x such that $\langle e_j, xe_k\rangle = 0$ for all $k \neq j \in V$ and $\mathcal{D}_{off}$ the operator space of off-diagonal operators namely the closed (in the norm, strong and weak* topologies) subspace of $x \in \mathbf{B}(\mathsf{h})$ such that $\langle e_k, xe_k\rangle = 0$ for all $k \in V$. Finally, we also denote by $(P_t)_{t\geq 0}$ the strongly continuous contraction semigroup on $\mathcal{B}(\mathsf{h})$ generated by G (see (10) and Theorem 3.1 of [6]). The diagonal algebra $\mathcal{D}$ is clearly isometrically isomorphic to the Banach space $l^\infty(V)$. Identifying $\mathcal{D}$ with $l^\infty(V)$ and taking the restrictions of $\mathcal{T}_t$ to $\mathcal{D}$ we find a weakly-* continuous classical sub-Markov semigroup $T = (T_t)_{t\geq 0}$ on $l^\infty(V)$. Its generator L is characterized (see [13] Lemma 2.19) by

$$Dom(L) = Dom(\mathcal{L}) \cap l^\infty(V), \quad Lf = \mathcal{L}(f) \quad \text{for all } f \in Dom(L).$$

A straightforward computation shows that the operator L satisfies

$$L_{jk} = \gamma_{jk}^-, \quad \text{for all } j,k \text{ with } \kappa_k < \kappa_j,$$

$$L_{kj} = \gamma_{kj}^{+}, \quad \text{for all } j,k \text{ with } \kappa_j < \kappa_k,$$
$$L_{kk} = -\sum_{\{k,j\in V|\kappa_k<\kappa_j\}} \gamma_{jk}^{-} - \sum_{\{k,j\in V|\kappa_j<\kappa_k\}} \gamma_{jk}^{+} = -(\mu_j + \lambda_j).$$

The following properties are important in this section.

Theorem 9. *Let $(\mathcal{T}_t)_{t\geq 0}$ be a generic QMS then:*

(a) The Abelian subalgebra $\mathcal{D}$ and the operator space $\mathcal{D}_{off}$ are $\mathcal{T}_t$-invariant for all $t \geq 0$. Moreover $\mathcal{T}_t(x) = P_t^* x P_t$ for all $x \in \mathcal{D}_{off}$.

(b) The spectral gap of a generic generator $\mathcal{L}$ is always equal to the spectral gap of the corresponding diagonal restriction L.

For a proof of the previous theorem, see Theorems 3 and 15 of [6].

Lemma 10. *For all selfadjoint $x \in \mathcal{D}_{off}$ and all $t \geq 0$ we have*

$$\|\delta_{nm}(P_t^* x P_t)\| \leq 2\sqrt{2}\, e^{-ct} \left(\|\delta_{nm}(x)\| + \|\delta_{nm'}(x)\|\right)$$

where

$$c = \frac{1}{2} \min \{ \lambda_n + \mu_n + \lambda_m + \mu_m \mid n \neq m \} \tag{11}$$

and $m' = m+1$ if $m+1 \neq n$ and $m' = m-1$ if $m+1 = n$.

Proof. Clearly

$$P_t^* x P_t = \sum_{j\neq k} e^{-t(\lambda_j+\mu_j+\lambda_k+\mu_k)/2 + it(\kappa_j-\kappa_k)} x_{jk} |e_j\rangle\langle e_k|.$$

In other words, the action of $P_t^* \cdot P_t$ on matrix elements x_{jk} of x corresponds to multiplication by a scalar. As a consequence, by Proposition 6, for all $n \neq m$, $\|\delta_{mn}(P_t^* x P_t)\|^2$ is smaller than

$$4e^{-2ct}\left(\left|x_{nm}e^{it(\kappa_n-\kappa_m)} - x_{mn}e^{-it(\kappa_n-\kappa_m)}\right|^2 + \max\left\{\sum_{i\neq m,n} |x_{ni}|^2, \sum_{i\neq m,n} |x_{mi}|^2\right\}\right)$$

Unfortunately

$$\left|x_{nm}e^{it(\kappa_n-\kappa_m)} - x_{mn}e^{-it(\kappa_n-\kappa_m)}\right|^2 \tag{12}$$

is not dominated by any multiple of $|x_{nm} - x_{mn}|^2$ (this happens, for instance, when $x_{nm} = x_{mn} \in \mathbb{R}$), therefore we bring into action another derivation $\delta_{nm'}$ where, for instance $m' = m+1$ if $m+1 \neq n$ and $m' = m-1$ if $m+1 = n$.

Since x is self-adjoint, by Proposition 7, we have

$$\begin{aligned}\left|x_{nm}e^{it(\kappa_n-\kappa_m)}-x_{mn}e^{-it(\kappa_n-\kappa_m)}\right|^2 &= 2\left|\Im(x_{nm}e^{it(\kappa_n-\kappa_m)})\right|^2\\ &\leq 2|x_{nm}|^2\leq 2d(n,m')^2\left\|\delta_{nm'}(x)\right\|^2\end{aligned}$$

and the $\max\{\cdot,\cdot\}$ term is dominated by $\|\delta_{nm}(x)\|^2$. The estimate of the norm $\|\delta_{nm}(P_t^*xP_t)\|$ now follows from the elementary inequality $(r+s)^{1/2}\leq r^{1/2}+s^{1/2}$ for all $r,s\geq 0$. □

Lemma 11. *For all selfadjoint $x\in\mathcal{D}_{off}$ and all $t\geq 0$ we have*

$$\|P_t^*xP_t\|_{LIP_d}\leq 2\sqrt{2}\,(2+r)\,e^{-ct}\,\|x\|_{LIP_d}$$

where c is given by (11) *and*

$$r=\sup_{n\neq m}\max\left\{\frac{d(m,m+1)}{d(n,m)},\frac{d(m,m-1)}{d(n,m)}\right\}$$

Proof. Note that, for all $n\neq m$, we have

$$\begin{aligned}\frac{\|\delta_{nm}(P_t^*xP_t)\|}{d(n,m)} &\leq 2\sqrt{2}\,e^{-ct}\left(\frac{\|\delta_{nm}(x)\|}{d(n,m)}+\frac{d(n,m')}{d(n,m)}\frac{\|\delta_{nm'}(x)\|}{d(n,m')}\right)\\ &\leq 2\sqrt{2}\,e^{-ct}\left(1+\frac{d(n,m')}{d(n,m)}\right)\|x\|_{LIP_d}\\ &\leq 2\sqrt{2}\,e^{-ct}\left(1+\frac{d(n,m)+d(m,m')}{d(n,m)}\right)\|x\|_{LIP_d}\\ &\leq 2\sqrt{2}\,e^{-ct}\left(2+\frac{d(m,m')}{d(n,m)}\right)\|x\|_{LIP_d}\,.\end{aligned}$$

The conclusion is now immediate.

□

Remark 5. It is worth noticing here that if the set $V=\mathbb{N}$ and the distance is $d(n,m)=|n-m|$, then $r=1$.

Given the structure of generic QMS $\mathcal{T}$ is clear that $\mathcal{T}$ restricted to $\mathcal{D}$ defines a classical semigroup T_t satisfying an inequality with classical Wasserstein curvature σ_d Moreover:

Proposition 12. *Let $\mathcal{T}$ be a generic QMS and let*

$$k:=\min_{n\neq m}\left\{\frac{\mu_n+\lambda_n+\mu_m+\lambda_m}{2}\wedge\sigma_d\right\}.$$

Suppose that

$$r = \sup_{n\neq m} \max\left\{\frac{d(m,m+1)}{d(n,m)}, \frac{d(m,m-1)}{d(n,m)}\right\} < \infty$$

then

$$\|\mathcal{T}_t(x)\|_{LIP_d} \leq (4\sqrt{2}+1+2\sqrt{2}\,r)e^{-tk}\|x\|_{LIP_d}$$

for all $t > 0$ *and for all* $x \in \mathbf{B}(\mathsf{h})$, x *selfadjoint.*

Proof. Let $\mathcal{E} : \mathbf{B}(\mathsf{h}) \mapsto \mathcal{D}$, where $\mathcal{D}$ is the diagonal subalgebra and $\mathcal{E}^{\perp} := I - \mathcal{E} : \mathbf{B}(\mathsf{h}) \mapsto \mathcal{D}_{off}$ then, by Lemma 11 and propostion 4, we obtain that

$$\begin{aligned}
\|\mathcal{T}_t(x)\|_{LIP_d} &\leq \|\mathcal{T}_t(\mathcal{E}(x))\|_{LIP_d} + \|\mathcal{T}_t(\mathcal{E}^{\perp}(x))\|_{LIP_d}\\
&\leq e^{-\sigma_d t}\|\mathcal{E}(x)\|_{LIP_d} + \|P_t^*\mathcal{E}^{\perp}(x)P_t\|_{LIP_d}\\
&\leq e^{-\sigma_d t}\|x\|_{LIP_d} + 2\sqrt{2}(2+r)e^{-ct}\|\mathcal{E}^{\perp}x\|_{LIP_d}\\
&\leq e^{-\sigma_d t}\|x\|_{LIP_d} + 2\sqrt{2}(2+r)e^{-ct}(\|\mathcal{E}(x)\|_{LIP_d} + \|x\|_{LIP_d})\\
&\leq (4\sqrt{2}+1+2\sqrt{2}\,r)e^{-(\sigma_d\wedge c)t}\|x\|_{LIP_d}
\end{aligned}$$

with $c = \frac{1}{2}\min\{\lambda_n + \mu_n + \lambda_m + \mu_m | n \neq m\}$.

□

Using the las sentence it follows that if

$$k := \min_{n\neq m}\left\{\frac{\mu_n + \lambda_n + \mu_m + \lambda_m}{2} \wedge \sigma_d\right\}.$$

and

$$r = \sup_{n\neq m} \max\left\{\frac{d(m,m+1)}{d(n,m)}, \frac{d(m,m-1)}{d(n,m)}\right\} < \infty$$

then taking $y = \frac{x}{(4\sqrt{2}+1+2\sqrt{2}\,r)e^{-tk}}$ we obtaint that

$$\begin{aligned}
&W_d(\mathcal{T}_{*t}(\rho_1), \mathcal{T}_{*t}(\rho_2))\\
&= (4\sqrt{2}+1+2\sqrt{2}\,r)e^{-tk} \sup_{\|x\|_{LIP_d}\leq 1} tr(\rho_1-\rho_2)\mathcal{T}_t\,(y)\\
&\leq (4\sqrt{2}+1+2\sqrt{2}\,r)e^{-tk} \sup_{\|\mathcal{T}_t(y)\|_{LIP_d}} tr(\rho_1-\rho_2)\mathcal{T}_t\,(y)\\
&\leq (4\sqrt{2}+1+2\sqrt{2}\,r)e^{-tk} \sup_{\|x\|_{LIP_d}\leq 1} tr(\rho_1-\rho_2)\mathcal{T}_t\,(x)\\
&= (4\sqrt{2}+1+2\sqrt{2}\,r)e^{-tk}W_d(\rho_1,\rho_2),
\end{aligned}$$

i.e., for all ρ_1, ρ_2 states in $\mathbf{B}(\mathsf{h})$ and for all $t > 0$

$$W_d(\mathcal{T}_{*t}(\rho_1), \mathcal{T}_{*t}(\rho_2)) \leq (4\sqrt{2} + 1 + 2\sqrt{2}\,r)e^{-tk}W_d(\rho_1, \rho_2) \qquad (13)$$

i.e., in other words, we obtain the following corollary

Corollary 13. *Let $\mathcal{T}$ be a generic QMS and let*

$$k := \min_{n \neq m} \left\{ \frac{\mu_n + \lambda_n + \mu_m + \lambda_m}{2} \wedge \sigma_d \right\}.$$

Suppose that

$$r = \sup_{n \neq m} \max \left\{ \frac{d(m, m+1)}{d(n, m)}, \frac{d(m, m-1)}{d(n, m)} \right\} < \infty$$

then

$$W_d(\mathcal{T}_{*t}(\rho_1), \mathcal{T}_{*t}(\rho_2)) \leq (4\sqrt{2} + 1 + 2\sqrt{2}\,r)e^{-tk}W_d(\rho_1, \rho_2)$$

for all states ρ_1, ρ_2.

Acknowledgments

The author would like to thank F. Fagnola for important discussions during the preparation of this work. The author is grateful to R. Quezada and the organizing committee for the invitation to the "BIRS Workshop (15w5086) Quantum Markov Semigroups in Analysis, Physics and Probability" and R. Rebolledo, for the invitation to the Congress "Stochastic Analysis and Mathematical Physics: the Biological challenge (SAMP2015)" where a part of this work was done. J. Agredo and Universidad Nacional de Colombia want to highlight the financial support from "Patrimonio autónomo fondo nacional de financiamiento para la ciencia, tecnología y la innovación, Francisco Jose de Caldas" and "Universidad Nacional de Colombia (UN)".

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