



The Decoherence-Free Subalgebra of Gaussian Quantum Markov Semigroups

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Abstract. We demonstrate a method for finding the decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ of a Gaussian quantum Markov semigroup on the von Neumann algebra $\mathcal{B}(\Gamma(\mathbb{C}^d))$ of all bounded operator on the Fock space $\Gamma(\mathbb{C}^d)$ on \mathbb{C}^d . We show that $\mathcal{N}(\mathcal{T})$ is a type I von Neumann algebra $L^\infty(\mathbb{R}^{d_c}; \mathbb{C}) \overline{\otimes} \mathcal{B}(\Gamma(\mathbb{C}^{d_f}))$ determined, up to unitary equivalence, by two natural numbers $d_c, d_f \leq d$. This result is illustrated by some applications and examples.

Keywords. Quantum Markov semigroup, Gaussian, Decoherence-free subalgebra, Araki's duality theorem, Symplectic spaces.

1. Introduction

Quantum channels and quantum Markov semigroups (QMS) describe the evolution of an open quantum system subject to noise because of the interaction with the surrounding environment. Couplings to external degrees of freedom typically lead to decoherence. Gaussian quantum channels and Markov semigroups play a key role because several models are based on linear couplings of bosonic systems to other bosonic systems with quadratic Hamiltonians. As a result, the time evolution is then determined by a Gaussian channel in the discrete time case and Markov semigroup in the time continuous case.

Decoherence-free subalgebras determine observables whose evolution is not affected by noise and play a fundamental role not only in the analysis of decoherence (see [2, 4, 7, 22, 26, 27] and the references therein) but also in the study of the structure of QMSs (see [18]).

The case of a norm-continuous QMS $(\mathcal{T}_t)_{t \geq 0}$ has been extensively studied ([13, 18] and the references therein). The generator is represented in a GKLS form $\mathcal{L}(x) = i[H, x] - (1/2) \sum_\ell (L_\ell^* L_\ell x - 2L_\ell^* L_\ell x + x L_\ell^* L_\ell)$ for some bounded operators L_ℓ, H with H self-adjoint and the decoherence-free subalgebra is characterized as the commutant of operators L_ℓ, L_ℓ^* and their iterated commutators with H ([13] Proposition 2.3).

Gaussian QMSs arise in several relevant models and form a class with a rich structure with a number of explicit formulas ([1, 10, 20, 31]), yet they are not norm-continuous. However, it is known that their generators can be written in a generalized GKLS form with operators L_ℓ that are linear (see (2.7)) and H quadratic (see (2.6)) in boson creation and annihilation operators a_j, a_k^\dagger ([11, 29]). As a consequence, operators L_ℓ, L_ℓ^* and their iterated generalized commutators with H are linear in boson creation and annihilation operators a_j, a_k^\dagger .

In this paper we consider Gaussian QMS on the von Neumann algebra $\mathcal{B}(\Gamma(\mathbb{C}^d))$ of all bounded operators on the Boson Fock space on \mathbb{C}^d and characterize their decoherence-free subalgebras as generalized commutants of these iterated generalized commutators (Theorem 3.2). Indeed, we show that it suffices to consider iterated commutators up to the order $2d - 1$. Moreover, we prove (Theorem 3.7) that the decoherence-free subalgebra is a type I von Neumann algebra $L^\infty(\mathbb{R}^{d_c}; \mathbb{C}) \otimes \mathcal{B}(\Gamma(\mathbb{C}^{d_f}))$ determined, up to unitary equivalence, by two natural numbers $d_c, d_f \leq d$. This conclusion is illustrated by some examples and a detailed analysis of the case of a Gaussian QMS with a single operator L_ℓ .

The symplectic structure on \mathbb{C}^d plays a fundamental role in the origin of von Neumann algebras of the type $L^\infty(\mathbb{R}^{d_c}; \mathbb{C})$ as possible decoherence-free subalgebras determined by generalized commutants via Araki's duality theorem. Moreover, even if at a purely algebraic level Theorem 3.2 looks like a natural generalization of the norm continuous case, several difficulties arise from unboundedness of operators L_ℓ and H and, as a consequence, unboundedness of the generator \mathcal{L} of the QMS \mathcal{T} . The defining property (3.1) of $\mathcal{N}(\mathcal{T})$ involves an operator x and the product x^*x , but, even if x belongs to the domain of the generator of the QMS \mathcal{T} , there is no reason why x^*x should (see e.g. [17]) therefore one has to work with quadratic forms. Domain problems arising from generalized commutators have to be carefully handled because one needs to make sense of generalized commutations such as $x[H, L_\ell] \subseteq [H, L_\ell]x$ and $e^{itH}xe^{-itH}L_\ell \subseteq L_\ell e^{itH}xe^{-itH}$ for $x \in \mathcal{N}(\mathcal{T})$.

The decoherence-free subalgebra of a QMS with unbounded generator was also characterized in [14]. In that paper the result was obtained through several technical assumptions (that hold in the case of Gaussian semigroups) and using a dilation of the QMS via quantum stochastic calculus. The proof we give here (Appendix B) is simpler because it does not appeal to these assumptions and is more direct because it does not use quantum stochastic calculus.

The paper is organized as follows. In Sect. 2 we introduce Gaussian QMS, present their construction by the minimal semigroup method (see [16]), prove well-definedness (conservativity or identity preservation) applying the sufficient condition of [9] and show the explicit formula for the action on Weyl operators (Theorem 2.4). Proofs, that can be obtained from applications of standard methods, are collected in Appendix A. Section 3 contains the main results of the paper. We first recall the definition of decoherence-free subalgebra. Then we prove its characterization for a Gaussian QMS (Theorem 3.2 with proof in Appendix B). Finally we prove the structure result Theorem 3.7. Applications and examples are presented in Sect. 4.

2. Gaussian QMSs

In this section we introduce our class of Gaussian semigroups starting from their generators and fix some notation. Let \mathfrak{h} be the Fock space $\mathfrak{h} = \Gamma(\mathbb{C}^d)$ which is isometrically isomorphic to $\Gamma(\mathbb{C}) \otimes \cdots \otimes \Gamma(\mathbb{C})$ with canonical orthonormal basis $(e(n_1, \dots, n_d))_{n_1, \dots, n_d \geq 0}$ (with $e(n_1, \dots, n_d) = e_{n_1} \otimes \cdots \otimes e_{n_d}$). Let a_j, a_j^\dagger be the creation and annihilation operator of the Fock representation of the d -dimensional Canonical Commutation Relations (CCR)

$$\begin{aligned} a_j e(n_1, \dots, n_d) &= \sqrt{n_j} e(n_1, \dots, n_{j-1}, n_j - 1, \dots, n_d), \\ a_j^\dagger e(n_1, \dots, n_d) &= \sqrt{n_j + 1} e(n_1, \dots, n_{j-1}, n_j + 1, \dots, n_d), \end{aligned}$$

The CCRs are written as $[a_j, a_k^\dagger] = \delta_{jk} \mathbb{1}$, where $[\cdot, \cdot]$ denotes the commutator, or, more precisely, $[a_j, a_k^\dagger] \subseteq \delta_{jk} \mathbb{1}$ because the domain of the operator in the left-hand side is smaller.

For any $g \in \mathbb{C}^d$, define the exponential vector $e(g)$ associated with g by

$$e_g = \sum_{n \in \mathbb{N}^d} \frac{g_1^{n_1} \cdots g_d^{n_d}}{\sqrt{n_1! \cdots n_d!}} e(n_1, \dots, n_d)$$

Creation and annihilation operators with test vector $v \in \mathbb{C}^d$ can also be defined on the total set of exponential vectors (see [28]) by

$$a(v)e_g = \langle v, g \rangle e_g, \quad a^\dagger(v)e_g = \frac{d}{d\varepsilon} e_{g+\varepsilon v} \Big|_{\varepsilon=0}$$

for all $u \in \mathbb{C}^d$. The unitary correspondence $\Gamma(\mathbb{C}^d) \mapsto \Gamma(\mathbb{C}) \otimes \cdots \otimes \Gamma(\mathbb{C})$

$$e_g \mapsto \sum_{n_1 \geq 0, \dots, n_d \geq 0} \frac{g_1^{n_1} \cdots g_d^{n_d}}{\sqrt{n_1! \cdots n_d!}} e_{n_1} \otimes \cdots \otimes e_{n_d}$$

allows one to establish the identities

$$a(v) = \sum_{j=1}^d \bar{v}_j a_j, \quad a^\dagger(u) = \sum_{j=1}^d u_j a_j^\dagger$$

for all $u^T = [u_1, \dots, u_d], v^T = [v_1, \dots, v_d] \in \mathbb{C}^d$.

The above operators are obviously defined on the linear manifold D spanned by the elements $(e(n_1, \dots, n_d))_{n_1, \dots, n_d \geq 0}$ of the canonical orthonormal basis of \mathfrak{h} that turns out to be an essential domain for all the operators considered so far. This also happens for quadrature operators

$$q(u) = (a(u) + a^\dagger(u)) / \sqrt{2} \quad u \in \mathbb{C}^d \tag{2.1}$$

that are symmetric and essentially self-adjoint on the domain D by Nelson's theorem on analytic vectors ([30] Th. X.39 p. 202). The linear span of exponential vectors also turns out to be an essential domain for operators $q(u)$ for the same reason. If the vector u has real (resp. purely imaginary) components one finds position (resp. momentum) operators and the commutation relation $[q(u), q(v)] \subseteq i\Im \langle u, v \rangle \mathbb{1}$ (where \Im and \Re denote the imaginary and real part of a complex number). Momentum operators, i.e. quadratures $q(ir)$ with $r \in \mathbb{R}^d$ are also denoted by $p(r) = \sum_{1 \leq j \leq d} r_j p_j$

where $p_j = i(a_j^\dagger - a_j)/\sqrt{2}$. In a similar way we write $q(r) = \sum_{1 \leq j \leq d} r_j q_j$ with $q_j = q(e_j) = (a_j^\dagger + a_j)/\sqrt{2}$.

Another set of operators that will play an important role in this paper are the Weyl operators, defined on the exponential vectors via the formula

$$W(z)e_g = e^{-\|z\|^2/2 - \langle z, g \rangle} e_{z+g} \quad z, g \in \mathbb{C}^d.$$

By this definition $\langle W(z)e_f, W(z)e_g \rangle = \langle e_f, e_g \rangle$ for all $f, g \in \mathbb{C}^d$, therefore $W(z)$ extends uniquely to a unitary operator on \mathfrak{h} . Weyl operators satisfy the CCR in the exponential form, namely, for every $z, z' \in \mathbb{C}^d$,

$$W(z)W(z') = e^{-i\Im \langle z, z' \rangle} W(z + z'). \tag{2.2}$$

It is well-known that $W(z)$ is the exponential of the anti self-adjoint operator $-i\sqrt{2}q(iz)$

$$W(z) = e^{-i\sqrt{2}q(iz)} = e^{za^\dagger - \bar{z}a}. \tag{2.3}$$

Finally, we recall here two relevant properties that are valid on D and on suitable dense domains

$$[a(v), W(z)] = \langle v, z \rangle W(z), \quad [a^\dagger(v), W(z)] = \langle z, v \rangle W(z). \tag{2.4}$$

A QMS $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$ is a weakly*-continuous semigroup of completely positive, identity preserving, weakly*-continuous maps on $\mathcal{B}(\mathfrak{h})$. The predual semigroup $\mathcal{T}_* = (\mathcal{T}_{*t})_{t \geq 0}$ on the predual space of trace class operators on \mathfrak{h} is a strongly continuous contraction semigroup.

Gaussian QMSs can be defined in various equivalent ways. Here we introduced them through their generator because it is the object we are mostly concerned with. The pre-generator, or form generator, of a Gaussian QMSs can be represented in a generalized (since operators L_ℓ, H are unbounded) Gorini–Kossakowski–Lindblad–Sudarshan (GKLS) form (see [29] Theorems 5.1, 5.2 and also [11, 32])

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_{\ell=1}^m (L_\ell^* L_\ell x - 2L_\ell^* x L_\ell + x L_\ell^* L_\ell). \tag{2.5}$$

where $1 \leq m \leq 2d$, and

$$H = \sum_{j,k=1}^d \left(\Omega_{jk} a_j^\dagger a_k + \frac{\kappa_{jk}}{2} a_j^\dagger a_k^\dagger + \frac{\bar{\kappa}_{jk}}{2} a_j a_k \right) + \sum_{j=1}^d \left(\frac{\zeta_j}{2} a_j^\dagger + \frac{\bar{\zeta}_j}{2} a_j \right), \tag{2.6}$$

$$L_\ell = \sum_{k=1}^d \left(\overline{v_{\ell k}} a_k + u_{\ell k} a_k^\dagger \right) = a(v_{\ell \bullet}) + a^\dagger(u_{\ell \bullet}), \tag{2.7}$$

$\Omega := (\Omega_{jk})_{1 \leq j, k \leq d} = \Omega^*$ and $\kappa := (\kappa_{jk})_{1 \leq j, k \leq d} = \kappa^T \in M_d(\mathbb{C})$, are $d \times d$ complex matrices with Ω Hermitian and κ symmetric, $V = (v_{\ell k})_{1 \leq \ell \leq m, 1 \leq k \leq d}$, $U = (u_{\ell k})_{1 \leq \ell \leq m, 1 \leq k \leq d} \in M_{m \times d}(\mathbb{C})$ are $m \times d$ matrices and $\zeta = (\zeta_j)_{1 \leq j \leq d} \in \mathbb{C}^d$. The notation $v_{\ell \bullet}$ and $u_{\ell \bullet}$ refers to vectors in \mathbb{C}^d obtained from the ℓ -th row of the corresponding matrices.

We exclude the case where the pre-generator \mathcal{L} reduces to the Hamiltonian part $i[H, x]$ and so we suppose that one among matrices V, U is non-zero. An application of Nelson’s theorem on analytic vectors ([30] Th. X.39 p. 202) shows that H , as

an operator with domain D , is essentially selfadjoint. In addition, operators L_ℓ are closable therefore we will identify them with their closure.

It can be shown (see [29] Theorems 5.1, 5.2) that a QMS \mathcal{T} is Gaussian if maps \mathcal{T}_{*t} of the predual semigroup \mathcal{T}_* preserve Gaussian states or, still in an equivalent way, maps \mathcal{T}_t act explicitly on Weyl operators (Theorem 2.4 below).

Clearly, \mathcal{L} is well defined on the dense (not closed) sub- $*$ -algebra of $\mathcal{B}(\mathfrak{h})$ generated by rank one operators $|\xi\rangle\langle\xi'|$ with $\xi, \xi' \in D$ because all operator compositions make sense. However, since the operators H, L_ℓ are unbounded, the domain of \mathcal{L} is not the whole of $\mathcal{B}(\mathfrak{h})$. For this reason we look at it as a pre-generator and describe in detail its extension to a generator of a QMS by the minimal semigroup method (Theorem 2.3 below).

Remark. The above generalized GKLS form is the most general with operators L_ℓ which are first order polynomials in a_j, a_j^\dagger and the self-adjoint operator H which is a second order polynomial in a_j, a_j^\dagger . Indeed, in the case where L_ℓ are as above plus a multiple of the identity operator, exploiting non uniqueness of GKLS representations (see [28], section 30) one can always apply a translation and reduce himself to the previous case.

We choose the minimum number of operators L_ℓ (also called Kraus operators), namely the parameter m .

Definition 2.1. A GKLS representation of \mathcal{L} is *minimal* if the number m in (2.5) is minimal.

A GKLS representation is minimal if and only if the following condition on V and U , that will be in force throughout the paper, holds.

Proposition 2.2. *The pre-generator \mathcal{L} has a minimal GKLS representation if and only if*

$$\ker(V^*) \cap \ker(U^T) = \{0\}. \tag{2.8}$$

Proof. A GKLS representation (2.5) is minimal GKLS if and only if the set $\{\mathbb{1}, L_1, \dots, L_m\}$ consists of linearly independent operators (see [28], Theorem 30.16), namely, $\alpha_0 \mathbb{1} + \sum_{\ell=1}^m \alpha_\ell L_\ell = 0$ for $\alpha_0, \alpha_\ell \in \mathbb{C}$ implies $\alpha_\ell = 0$ for $\ell = 0, 1, \dots, m$. This identity is equivalent to

$$\alpha_0 \mathbb{1} + \sum_{j=1}^d (V^* \alpha)_j a_j + \sum_{j=1}^d (U^T \alpha)_j a_j^\dagger = 0.$$

Since $\{\mathbb{1}, a_1, a_1^\dagger, \dots, a_d, a_d^\dagger\}$ is a linearly independent set, the last equation is equivalent to $\alpha \in \ker(V^*), \alpha \in \ker(U^T), \alpha_0 = 0$ and the proof is complete. \square

For all $x \in \mathcal{B}(\mathfrak{h})$ consider the quadratic form with domain $D \times D$

$$\begin{aligned} \mathcal{L}(x)[\xi', \xi] &= i \langle H \xi', x \xi \rangle - i \langle \xi', x H \xi \rangle \\ &\quad - \frac{1}{2} \sum_{\ell=1}^m (\langle \xi', x L_\ell^* L_\ell \xi \rangle - 2 \langle L_\ell \xi', x L_\ell \xi \rangle + \langle L_\ell^* L_\ell \xi', x \xi \rangle) \end{aligned} \tag{2.9}$$

We postpone to the Appendix the construction of the unique Gaussian QMS with pre-generator (2.5) and state here the final result.

Theorem 2.3. *There exists a unique QMS, $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$ such that, for all $x \in \mathcal{B}(\mathfrak{h})$ and $\xi, \xi' \in D$, the function $t \mapsto \langle \xi', \mathcal{T}_t(x)\xi \rangle$ is differentiable and*

$$\frac{d}{dt} \langle \xi', \mathcal{T}_t(x)\xi \rangle = \mathfrak{L}(\mathcal{T}_t(x))[\xi', \xi] \quad \forall t \geq 0.$$

The domain of the generator consists of $x \in \mathcal{B}(\mathfrak{h})$ for which the quadratic form $\mathfrak{L}(x)$ is represented by a bounded operator.

Weyl operators do not belong to the domain of the generator of \mathcal{T} because a straightforward computation (see, for instance, Appendix A Sect. 4.5) shows that the quadratic form $\mathfrak{L}(x)$ is unbounded. In spite of this we have the following explicit formula (see [11, 32])

Theorem 2.4. *Let $(\mathcal{T}_t)_{t \geq 0}$ be the quantum Markov semigroup with generalized GKLS generator associated with H, L_ℓ as above. For all Weyl operator $W(z)$ we have*

$$\mathcal{T}_t(W(z)) = \exp\left(-\frac{1}{2} \int_0^t \Re \langle e^{sZ} z, C e^{sZ} z \rangle ds + i \int_0^t \Re \langle \zeta, e^{sZ} z \rangle ds\right) W(e^{tZ} z) \quad (2.10)$$

where the real linear operators Z, C on \mathbb{C}^d are

$$Zz = \left[\frac{(\overline{U^*U} - V^*V)}{2} + i\Omega \right] z + \left[(U^T V - V^T U) / 2 + i\kappa \right] \bar{z} \quad (2.11)$$

$$Cz = (\overline{U^*U} + V^*V) z + (U^T V + V^T U) \bar{z} \quad (2.12)$$

We refer to Sect. 4.5 for the proof.

3. Structure of $\mathcal{N}(\mathcal{T})$

The decoherence-free subalgebra (see [2, 7, 13, 14]) of \mathcal{T} is defined as

$$\mathcal{N}(\mathcal{T}) = \{x \in \mathcal{B}(\mathfrak{h}) \mid \mathcal{T}_t(x^*x) = \mathcal{T}_t(x^*)\mathcal{T}_t(x), \mathcal{T}_t(xx^*) = \mathcal{T}_t(x)\mathcal{T}_t(x^*), \forall t \geq 0\}. \quad (3.1)$$

This is the biggest sub von Neumann algebra of $\mathcal{B}(\mathfrak{h})$ on which maps \mathcal{T}_t act as *-homomorphisms by the following known facts (see e.g. Evans [15] Th. 3.1).

Proposition 3.1. *Let \mathcal{T} be a QMS on $\mathcal{B}(\mathfrak{h})$ and let $\mathcal{N}(\mathcal{T})$ be the set defined by (3.1). Then:*

1. $\mathcal{N}(\mathcal{T})$ is \mathcal{T}_t -invariant for all $t \geq 0$,
2. For all $x \in \mathcal{N}(\mathcal{T})$ and all $y \in \mathcal{B}(\mathfrak{h})$ we have $\mathcal{T}_t(x^*y) = \mathcal{T}_t(x^*)\mathcal{T}_t(y)$ and $\mathcal{T}_t(y^*x) = \mathcal{T}_t(y^*)\mathcal{T}_t(x)$,
3. $\mathcal{N}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathfrak{h})$.

The decoherence-free subalgebra of a QMS with a bounded generator, i.e. written in a GKLS form with bounded operators H, L_ℓ instead of (2.7), (2.6) is the commutator of the set of operators $\delta_H^n(L_\ell), \delta_H^n(L_\ell^*)$ with $\ell = 1, \dots, m, n \geq 0$ where $\delta_H(x) = [H, x]$.

Generators of Gaussian QMSs are represented in a generalized GKLS form with unbounded operators L_ℓ, H , but $\mathcal{N}(\mathcal{T})$ can be characterized in a similar way

considering *generalized* commutator of a set of unbounded operators. We recall that, the generalized commutator of an unbounded operator L is the set of bounded operators x for which $xL \subseteq Lx$, namely Lx is an extension of xL .

We begin our investigation on $\mathcal{N}(\mathcal{T})$ by the following

Theorem 3.2. *The decoherence-free subalgebra of a Gaussian QMS with generator in a generalized GKLS form associated with operators L_ℓ, H as in (2.7), (2.6) is the generalized commutator of linear combinations of operators*

$$\delta_H^n(L_\ell), \quad \delta_H^n(L_\ell^*) \quad \ell = 1, \dots, m, \quad 0 \leq n \leq 2d - 1 \tag{3.2}$$

where $\delta_H(x) = [H, x]$ denotes the generalized commutator and δ_H^n denotes its n -th iterate. Moreover $\mathcal{T}_t(x) = e^{itH} x e^{-itH}$ for all $t \geq 0$ and $x \in \mathcal{N}(\mathcal{T})$.

We defer the proof to Appendix B. Here we give an example to show that inequality $n \leq 2d - 1$ is sharp.

Example. Consider the Gaussian QMS with only one operator L_ℓ , i.e. $m = 1$ and

$$L_1 = p_1, \quad H = q_d^2 + \sum_{j=1}^{d-1} p_{j+1} q_j$$

Compute

$$\delta_H(L_1) = ip_2, \quad \delta_H^2(L_1) = -p_3, \dots, \quad \delta_H^{d-1}(L_1) = i^{d-1} p_d, \quad \delta_H^d(L_1) = i^d q_d$$

therefore $\delta_H^{d+1}(L_1) = i^{d+1} q_{d-1}$. Iterating commutators we see that $\delta_H^{d+k}(L_1)$ is proportional to q_{d-k} so that, for $k = d - 1$ one gets q_1 . Clearly, for all k with $0 \leq k \leq d - 1$

$$\left\{ \delta_H^j(L_1), \delta_H^j(L_1^*) \mid j \leq d + k \right\}'$$

which is isomorphic to $L^\infty(\mathbb{R}^{d-1-k}; \mathbb{C})$ (i.e. measurable functions of momentum operators p_1, \dots, p_{d-1+k}). Summing up, if we consider $2d - 1$ iterated commutators we get all p_j, q_j and $\mathcal{N}(\mathcal{T})$ is trivial by the irreducibility of the Weyl representation of CCR.

In the sequel we provide a simpler characterization of $\mathcal{N}(\mathcal{T})$ in terms of *real* subspaces of \mathbb{C}^d and find its structure. In order to make clear the thread of the discussion, we omit technicalities related with unbounded operators that can be easily fixed because D is an essential domain for operators involved in our computations and we concentrate on the algebraic aspect. A straightforward computation yields

$$[H, L_\ell] = \sum_{i=1}^d \left((\Omega u_{\ell\bullet} - \kappa \bar{v}_{\ell\bullet})_i a_i^\dagger - (\bar{\Omega} \bar{v}_{\ell\bullet} - \bar{\kappa} u_{\ell\bullet})_i a_i \right) + \frac{\langle \zeta, u_{\ell\bullet} \rangle - \langle v_{\ell\bullet}, \zeta \rangle}{2} \mathbb{1}.$$

Therefore the set of operators of which we have to consider the generalized commutator, thanks to the CCR, is particularly simple and contains only linear combinations of creation and annihilation operators together with a multiple of the identity $\mathbb{1}$ that plays no role.

Now notice that each linear combination of creation and annihilation operators is uniquely determined by a pair v, u of vectors in \mathbb{C}^d representing coefficients of

annihilation and creation operators so that, for example, the operator L_ℓ in (2.7) and its adjoint L_ℓ^* are determined by

$$L_\ell = \sum_{j=1}^d \left(\bar{v}_{\ell j} a_j + u_{\ell j} a_j^\dagger \right) \rightsquigarrow \begin{bmatrix} \bar{v}_{\ell \bullet} \\ u_{\ell \bullet} \end{bmatrix}, \quad L_\ell^* = \sum_{j=1}^d \left(\bar{u}_{\ell j} a_j + v_{\ell j} a_j^\dagger \right) \rightsquigarrow \begin{bmatrix} \bar{u}_{\ell \bullet} \\ v_{\ell \bullet} \end{bmatrix}$$

In a similar way, after computing commutators,

$$[H, L_\ell] \rightsquigarrow \begin{bmatrix} -\Omega^T & \bar{\kappa} \\ -\kappa & \Omega \end{bmatrix} \begin{bmatrix} \bar{v}_{\ell \bullet} \\ u_{\ell \bullet} \end{bmatrix}, \quad [H, L_\ell^*] \rightsquigarrow \begin{bmatrix} -\Omega^T & \bar{\kappa} \\ -\kappa & \Omega \end{bmatrix} \begin{bmatrix} \bar{u}_{\ell \bullet} \\ v_{\ell \bullet} \end{bmatrix}$$

Denote by \mathbb{H} the above $2d \times 2d$ matrix (built by four $d \times d$ matrices)

$$\mathbb{H} = \begin{bmatrix} -\Omega^T & \bar{\kappa} \\ -\kappa & \Omega \end{bmatrix}$$

and let \mathcal{V} be the real subspace of \mathbb{C}^{2d} generated by vectors

$$\mathbb{H}^n \begin{bmatrix} \bar{v}_{\ell \bullet} \\ u_{\ell \bullet} \end{bmatrix}, \quad \mathbb{H}^n \begin{bmatrix} \bar{u}_{\ell \bullet} \\ v_{\ell \bullet} \end{bmatrix} \tag{3.3}$$

with $\ell = 1, \dots, m$ and $0 \leq n \leq 2d - 1$.

The above remarks allow us to associate with elements of (3.2) a set of vectors in \mathbb{C}^{2d} and characterize the generalized commutant of (3.2) in a purely algebraic way.

Lemma 3.3. *An operator $x \in \mathcal{B}(\mathfrak{h})$ belongs to $\mathcal{N}(\mathcal{T})$ if and only if it belongs to the generalized commutant of*

$$\{ q(iw) \mid w \in \mathcal{M} \} \tag{3.4}$$

where

$$\mathcal{M} = \text{Lin}_{\mathbb{R}} \{ i(v + u), v - u \mid [\bar{v}, u]^T \in \mathcal{V} \} \subset \mathbb{C}^d. \tag{3.5}$$

Proof. By the above remarks we know that the operators in the set (3.2) are linear combination of annihilation and creation operators up to a multiple of the identity operator and the generalized commutant of (3.2) coincides with the generalized commutant of

$$\{ a(v) + a^\dagger(u) \mid [\bar{v}, u]^T \in \mathcal{V} \}. \tag{3.6}$$

To conclude the proof we just need to show that the commutants of (3.4) and (3.6) are the same. Notice at first that if $[\bar{v}, u]^T \in \mathcal{V}$ also $[\bar{u}, v]^T \in \mathcal{V}$, indeed if $\delta_H^n(L_\ell) = a(v) + a^\dagger(u)$ then $\delta_H^n(L_\ell^*) = (-1)^n \delta_H^n(L_\ell)^* = (-1)^n (a(u) + a^\dagger(v))$ on the domain D . Now from (2.1) we obtain

$$\begin{aligned} \sqrt{2}q(i(u - v)) &= i(a(v) + a^\dagger(u)) - i(a(u) + a^\dagger(v)), \\ \sqrt{2}(a(v) + a^\dagger(u)) &= q(v + u) - iq(i(u - v)). \end{aligned}$$

Therefore every element of (3.4) is a linear combination of elements of (3.6) and viceversa, concluding the proof. \square

In order to describe the structure of the decoherence-free subalgebra we recall now some useful definitions and properties of symplectic spaces. At first note that \mathbb{C}^d equipped with the real scalar product $\Re\langle \cdot, \cdot \rangle$ is a real Hilbert space. Considering instead $\Im\langle \cdot, \cdot \rangle$ we obtain a bilinear, antisymmetric (i.e. $\Im\langle z_1, z_2 \rangle = -\Im\langle z_2, z_1 \rangle$), for

all $z_1, z_2 \in \mathbb{C}^d$) and non-degenerate (i.e. $\Im\langle z_1, z_2 \rangle = 0$ for all $z_2 \in \mathbb{C}^d$ implies $z_1 = 0$) form also called a symplectic form. We now recall the following definitions.

Definition 3.4. Let $M \subset \mathbb{C}^d$ a real linear subspace.

1. M is a symplectic space if $\Im\langle \cdot, \cdot \rangle$ is non degenerate when restricted to elements of M .
2. Two elements z_1, z_2 of M are called symplectically orthogonal if they satisfy $\Im\langle z_1, z_2 \rangle = 0$.
3. Let $M_1 \subset M$ be a real linear subspace. We call symplectic complement of M_1 in M , and denote it by M_1' , the set

$$M' = \{z \in M \mid \Im\langle z, m \rangle = 0 \quad \forall m \in M_1\}.$$

4. M_1 is an isotropic subspace if $M_1 \subset M_1'$.
5. M_1 is a symplectic subspace if M_1 is a symplectic space (i.e. the symplectic form $\Im\langle \cdot, \cdot \rangle$ is non degenerate when restricted to elements of M_1).

In order to fix some of the concepts in the above definition we provide the following

Example. Consider $M = \mathbb{C}^d$ which is a symplectic space and let $(e_j)_{j=1}^d$ be its canonical complex orthonormal basis. Clearly $(e_j, ie_j)_{j=1}^d$ is an orthonormal basis for \mathbb{C}^d considered as a real Hilbert space. Consider now a vector e_j for a fixed index j . It is orthogonal to all other elements of the basis with respect to the real scalar product $\Re\langle \cdot, \cdot \rangle$, however it is symplectically orthogonal to all other elements of the basis except for ie_j .

Consider now $M_1 = \text{Lin}_{\mathbb{R}}\{e_1, ie_1\}$ and $M_2 = \text{Lin}_{\mathbb{R}}\{e_2\}$, which are real linear subspaces of \mathbb{C}^d . It is easy to see that

$$M_1' = \text{Lin}_{\mathbb{R}}\{e_2, ie_2, \dots, e_d, ie_d\}, \quad M_2' = \text{Lin}_{\mathbb{R}}\{e_1, ie_1, e_2, e_3, ie_3, \dots, e_d, ie_d\}.$$

In particular $M_2 \subset M_2'$ hence it is an isotropic subspace, while M_1 is a symplectic subspace.

Eventually it is worth noticing that not all symplectic subspaces of M are also complex subspaces; from here the need to consider real vector spaces. Indeed, consider $M_3 = \text{Lin}_{\mathbb{R}}\{e_1 + e_2, ie_1\}$. It is easy to prove that M_3 is a symplectic subspace of M but it is not a complex subspace, since $e_1 = (-i)ie_1 \notin M_3$.

The previous example although seemingly simple is actually quite representative of what happens in the general case. In analogy with classical linear algebra most complicated situations can be simplified performing a change of basis through a homomorphism. We provide here the analogous definition for symplectic spaces.

Definition 3.5. Let $M_1, M_2 \subset \mathbb{C}^d$ be symplectic spaces. We say $B : M_1 \rightarrow M_2$ is a symplectic transformation if it is a real linear map and moreover

$$\Im\langle Bz_1, Bz_2 \rangle = \Im\langle z_1, z_2 \rangle, \quad \forall z_1, z_2 \in M_1.$$

We say B is a Bogoliubov transformation or symplectomorphism if it is an invertible symplectic transformation.

Next Proposition collects all the properties of symplectic spaces we need (see [6] for a comprehensive treatment)

Proposition 3.6. *Let $M \subset \mathbb{C}^d$ be a symplectic space and let $(e_j)_{j=1}^d$ be the canonical complex orthonormal basis of \mathbb{C}^d .*

1. *There exists a symplectomorphism B*

$$B : M \rightarrow \text{Lin}_{\mathbb{R}}\{e_1, ie_1, \dots, e_{d_1}, ie_{d_1}\}$$

In particular $\dim_{\mathbb{R}} M = 2d_1$.

2. *If $M_1 \subset M$ is a real linear subspace of M then M_1 is also a symplectic subspace if and only if $M_1 \cap M_1' = \{0\}$.*
3. *If M_1 is an isotropic subspace with $d_1 = \dim_{\mathbb{R}} M_1$ then there exists a symplectomorphism B such that*

$$B : M_1 \rightarrow \text{Lin}_{\mathbb{R}}\{e_1, \dots, e_{d_1}\}.$$

We give a proof in Appendix C for self-containedness.

For all subset \mathcal{M} of \mathbb{C}^d we denote by $\mathcal{W}(\mathcal{M})$ the von Neumann algebra generated by Weyl operators $W(z)$ with $z \in \mathcal{M}$. H. Araki’s Theorem 4 p. 1358 in [3], sometimes referred to as duality for Bose fields, (see also [25] Theorem 1.3.2 (iv) for a proof with our notation), up to a constant in the symplectic form and also [21] Theorem 1.1) shows that the commutant of $\mathcal{W}(\mathcal{M})$ is $\mathcal{W}(\mathcal{M}')$. Applying this result we can prove the following

Theorem 3.7. *The decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ is the von Neumann subalgebra of $\mathcal{B}(\mathfrak{h})$ generated by Weyl operators $W(z)$ such that z belongs to the symplectic complement of (3.5). Moreover, up to unitary equivalence,*

$$\mathcal{N}(\mathcal{T}) = L^\infty(\mathbb{R}^{d_c}; \mathbb{C}) \overline{\otimes} \mathcal{B}(\Gamma(\mathbb{C}^{d_f})) \tag{3.7}$$

for a pair of natural numbers $d_c, d_f \leq d$.

The subscript f (resp. c) stands for full (resp. commutative).

Proof. By Lemma 3.3 any $x \in \mathcal{N}(\mathcal{T})$ satisfies $xq(iw) \subseteq q(iw)x$ for all $w \in \mathcal{M}$. Therefore, for all real number r , $x(\mathbb{1} + irq(iw)) \subseteq (\mathbb{1} + irq(iw))x$ and, right and left multiplying by the resolvent $(\mathbb{1} + irq(iw))^{-1}$ which is a bounded operator

$$(\mathbb{1} + irq(iw))^{-1}x = x(\mathbb{1} + irq(iw))^{-1}.$$

Iterating n times and considering $r = 1/n$ we find

$$(\mathbb{1} + iq(iw)/n)^{-n}x = x(\mathbb{1} + iq(iw)/n)^{-n}$$

and, taking the limit as n goes to $+\infty$, by the Hille-Yosida theorem ([5] Theorem 3.1.10 p.371) we have

$$\begin{aligned} W(w)x &= e^{-iq(iw)}x = \lim_{n \rightarrow \infty} (\mathbb{1} + iq(iw)/n)^{-n}x \\ &= x \lim_{n \rightarrow \infty} (\mathbb{1} + iq(iw)/n)^{-n} = xW(w) \end{aligned}$$

and so x belongs to $\mathcal{W}(\mathcal{M})'$ which coincides with $\mathcal{W}(\mathcal{M}')$ by Araki’s Theorem 4 in [3].

Conversely, if z belongs to the symplectic complement of \mathcal{M} , then from (2.4) and (2.1) we have $W(z)q(iw)e_g = q(iw)W(z)e_g$ for all $w \in \mathcal{M}$ and $g \in \mathbb{C}^d$. Since the linear span of exponential vectors is an essential domain for $q(iw)$, for all $\xi \in \text{Dom}(q(iw))$ there exists a sequence $(\xi_n)_{n \geq 1}$ in \mathcal{E} such that $(q(iw)\xi_n)_{n \geq 1}$ converges to $q(iw)\xi$. It follows that $(q(iw)W(z)\xi_n)_{n \geq 1}$ converges and, since $q(iw)$ is closed, $W(z)\xi$ belongs to $\text{Dom}(q(iw))$ and $W(z)q(iw)\xi = q(iw)W(z)\xi$, namely $q(iw)W(z)$ is an extension of $W(z)q(iw)$. Therefore $W(z)$ belongs to the generalized commutant of all $q(iw)$ with $w \in \mathcal{M}$ and therefore to $\mathcal{N}(\mathcal{T})$ by Lemma 3.3.

In order to prove (3.7) consider $\mathcal{M}^c := \mathcal{M} \cap \mathcal{M}'$ which is a real linear subspace of both \mathcal{M} and \mathcal{M}' . Consider now \mathcal{M}^r and \mathcal{M}^f as the real linear complement of \mathcal{M}^c in \mathcal{M} and \mathcal{M}' respectively, i.e.

$$\mathcal{M} = \mathcal{M}^c \oplus \mathcal{M}^r, \quad \mathcal{M}' = \mathcal{M}^c \oplus \mathcal{M}^f.$$

($\mathcal{M}^c \perp \mathcal{M}^r$ and $\mathcal{M}^c \perp \mathcal{M}^f$, more precisely, they are orthogonal with respect to the real part of the scalar product). We will show that \mathcal{M}^f is a symplectic subspace of \mathbb{C}^d and that it is symplectically orthogonal to both \mathcal{M}^c and \mathcal{M}^r . Suppose $z \in \mathcal{M}^f$ is such that $\Im\langle z, z_f \rangle = 0$ for all $z_f \in \mathcal{M}^f$. By construction $\Im\langle z, z_c \rangle = 0$ for all $z_c \in \mathcal{M}^c = \mathcal{M} \cap \mathcal{M}'$. Therefore

$$\Im\langle z, m \rangle = 0, \quad \forall m \in \mathcal{M}',$$

since $\mathcal{M}' = \mathcal{M}^c \oplus \mathcal{M}^f$. Therefore $z \in \mathcal{M}'' = \mathcal{M}$, but then

$$z \in \mathcal{M} \cap \mathcal{M}' \cap \mathcal{M}^f = \mathcal{M}^c \cap \mathcal{M}^f = \{0\}.$$

Hence \mathcal{M}^f is a symplectic subspace. Eventually, $\mathcal{M}^f \subset \mathcal{M}'$ and $\mathcal{M} = \mathcal{M}'' \subset (\mathcal{M}^f)'$. In particular \mathcal{M}^f is symplectically orthogonal to both $\mathcal{M}^r, \mathcal{M}^c$. Let $d_c = \dim_{\mathbb{R}} \mathcal{M}^c$ and $2d_f = \dim_{\mathbb{R}} \mathcal{M}^f$ which is even by Proposition 3.6 1. Still by Proposition 3.6 we can find a symplectomorphism B such that

$$B : \mathcal{M}' \rightarrow \text{Lin}_{\mathbb{R}}\{e_1, \dots, e_{d_c}\} \oplus \text{Lin}_{\mathbb{R}}\{e_{d_c+1}, ie_{d_c+1}, \dots, e_{d_c+d_f}, ie_{d_c+d_f}\},$$

where $(e_j)_{j=1}^{d_c+d_f}$ is the canonical complex orthonormal basis of $\mathbb{C}^{d_c+d_f}$. Eventually, since symplectic transformation in finite dimensional symplectic spaces are always implemented by unitary transformations on the Fock space (see [12] Theorem 3.8), we obtain the final result. \square

Remark. An analogous argument to the proof of the previous theorem allows us to show that also \mathcal{M}^r is a symplectic subspace which is symplectically orthogonal to both \mathcal{M}^c and \mathcal{M}^f . If $2d_r = \dim_{\mathbb{R}} \mathcal{M}^r$, in total analogy with the proof, we can always find a symplectomorphism such that

$$\mathcal{M} \cup \mathcal{M}' \simeq \mathbb{C}^{d_r} \oplus \mathcal{M}^c \oplus \mathbb{C}^{d_f},$$

where \mathcal{M}_c is the real subspace of \mathbb{C}^{d_c} generated by $\{e_1, \dots, e_{d_c}\}$. In particular, after the unitary transformation associated with the symplectic transformation, we have

$$\mathcal{W}(\mathcal{M} \cup \mathcal{M}') = \mathcal{B}(\Gamma(\mathbb{C}^{d_r})) \overline{\otimes} L^\infty(\mathbb{R}^{d_c}; \mathbb{C}) \overline{\otimes} \mathcal{B}(\Gamma(\mathbb{C}^{d_f})).$$

Remark. It is worth noticing here that a QMS with $\mathcal{N}(\mathcal{T})$ as in (3.7) does not necessarily admit a dilation with d_c classical noises because the corresponding Kraus

operators L_ℓ could be normal but not self-adjoint (see Sect. 4.2 for an example) and so one may find obstructions to dilations with classical noises as shown in [19].

Corollary 3.8. *The decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ is generated by Weyl operators $W(z)$ with z belonging to real subspaces of $\ker(C)$ that are Z -invariant.*

Proof. By Theorem 3.7 it suffices to show that z belongs to the symplectic complement \mathcal{M}' of (3.5) if and only if it belongs to a real subspace of $\ker(C)$ that is Z -invariant.

If z belongs to \mathcal{M}' then $W(z) \in \mathcal{N}(\mathcal{T})$ and $\mathcal{T}_t(W(z)) = e^{itH}W(z)e^{-itH}$ for all $t \geq 0$. Comparison with (2.10) yields

$$\begin{aligned} & e^{itH}W(z)e^{-itH} \\ &= \exp\left(-\frac{1}{2}\int_0^t \Re\langle e^{sZ}z, Ce^{sZ}z\rangle ds + i\int_0^t \Re\langle \zeta, e^{sZ}z\rangle ds\right)W(e^{tZ}z). \end{aligned}$$

Unitarity of both left and right operators implies $\Re\langle e^{sZ}z, Ce^{sZ}z\rangle = 0$ for all $s \geq 0$ and $e^{sZ}z$ belongs to $\ker(C)$ for all $s \geq 0$, namely, in an equivalent way, z and also Zz (by differentiation) belong to $\ker(C)$.

Conversely, if z belongs to a real subspace of $\ker(C)$ that is Z -invariant, then $e^{sZ}z$ also belongs to that subset for all $s \geq 0$. The explicit formula (2.10) shows that

$$\mathcal{T}_t(W(z)) = \exp\left(i\int_0^t \Re\langle \zeta, e^{sZ}z\rangle ds\right)W(e^{tZ}z)$$

therefore

$$\mathcal{T}_t(W(z)^*)\mathcal{T}_t(W(z)) = e^{itH}W(z)^*W(z)e^{-itH} = \mathbb{1} = \mathcal{T}(W(z)^*W(z))$$

and, in the same way, $\mathcal{T}_t(W(z))\mathcal{T}_t(W(z)^*) = \mathcal{T}(W(z)W(z)^*)$. It follows that $W(z) \in \mathcal{N}(\mathcal{T})$ and z belongs to the symplectic complement of (3.5) by Theorem 3.7. \square

The following corollary shows that we can perform a unitary transformation of the Fock space in order to reduce the number of creation and annihilation operators that appear in the Kraus' operators.

Corollary 3.9. *There exists a unitary transformation U of the Fock space such that*

$$UL_\ell U^* = \sum_{j=1}^{d_r+d_c} (\bar{v}_j a_j + u_j a_j^\dagger)$$

Proof. It suffices to consider the transformation obtained in the Remark after Theorem 3.7. Indeed each Kraus operator corresponds to a vector $[\bar{v}, u]^T \in \mathcal{V}$ which in turn corresponds to two generators in the subspace \mathcal{M} . Performing the symplectomorphism in the cited Remark we have

$$\mathcal{M} \simeq \mathbb{C}^{d_r} \oplus \mathcal{M}^c$$

which has dimension $2d_r + d_c$. In particular if U is the unitary transformation that implements this symplectomorphism $UL_\ell U^*$ will depend at most from $d_r + d_c$ modes. \square

Example. One may wonder if H can also be written in a special form in the new representation of the CCR, for example as the sum of two self-adjoint operators, one depending only on $b_1, b_1^\dagger, \dots, b_{d_r+d_c}, b_{d_r+d_c}^\dagger$ and the other depending only on $b_{d_r+d_c+1}, b_{d_r+d_c+1}^\dagger, \dots, b_d, b_d^\dagger$. This happens when $\mathcal{N}(\mathcal{T})$ is a countable sum of type I factors (see [13]) but not in the case of Gaussian QMSs with $d_c > 0$ as shows this example.

Let $d = 2$, $m = 1$ and

$$L = q_1, \quad H = q_1 p_2.$$

Clearly, by Theorem 3.2, $\mathcal{N}(\mathcal{T})$ is the algebra $L^\infty(\mathbb{R}; \mathbb{C}) \overline{\otimes} \mathcal{B}(\Gamma(\mathbb{C}))$ but H is the product of two operators depending on different coordinates.

4. Applications

In this section we present two examples to illustrate the admissible structures of decoherence-free subalgebras of a Gaussian QMS on $\mathcal{B}(\Gamma(\mathbb{C}^d))$ with $d \geq 2$ and the application to an open system of two bosons in a common environment (see Ref. [8]). We begin by considering the case of only one noise operator.

4.1. The Case One $L, H = N$

The operators (2.7) and (2.6) are the closure of operators defined on D

$$L = \sum_{j=1}^d (\bar{v}_j a_j + u_j a_j^\dagger), \quad H = \sum_{j=1}^d a_j^\dagger a_j \quad (4.1)$$

(either v or u is nonzero). We compute recursively

$$\delta_H^{2n+1}(L) = \sum_{j=1}^d (u_j a_j^\dagger - \bar{v}_j a_j), \quad \delta_H^{2n}(L) = L$$

for all $n \geq 0$, and, in the same way, $\delta_H^{2n+1}(L^*) = \sum_{j=1}^d (v_j a_j^\dagger - \bar{u}_j a_j)$, $\delta_H^{2n}(L^*) = L^*$. It follows that \mathcal{M}' is the symplectic complement of

$$\text{Lin}_{\mathbb{R}} \{ v - u, v + u, i(v + u), i(v - u) \} = \text{Lin}_{\mathbb{R}} \{ v, u, iv, iu \} = \text{Lin}_{\mathbb{C}} \{ v, u \}.$$

Thus \mathcal{M}' is the orthogonal (for the complex scalar product) of the complex linear subspace generated by v and u , it is a complex subspace of \mathbb{C}^d and

$$\mathcal{N}(\mathcal{T}) = \{ W(z) \mid z \in \mathcal{M}' \} = \mathcal{B}(\Gamma(\mathcal{M}')).$$

If v, u are linearly independent, then the complex dimension of \mathcal{M}' is $d - 2$, and $\mathcal{N}(\mathcal{T})$ is isomorphic to $\mathcal{B}(\Gamma(\mathbb{C}^{d-2}))$.

TABLE 1 $\mathcal{N}(\mathcal{T})$ that can arise with one L and $H = 0$

	$\dim_{\mathbb{R}} \mathcal{M}$	d_c	d_r	d_f	$\mathcal{N}(\mathcal{T})$	L
1 st	1	1	0	$d - 1$	$L^\infty(\mathbb{R}; \mathbb{C}) \overline{\otimes} \mathcal{B}(\Gamma(\mathbb{C}^{d-1}))$	$L = q_1$
2 nd	2	2	0	$d - 2$	$L^\infty(\mathbb{R}^2; \mathbb{C}) \overline{\otimes} \mathcal{B}(\Gamma(\mathbb{C}^{d-2}))$	$L = q_1 + iq_2$
3 rd	2	0	1	$d - 1$	$\mathcal{B}(\Gamma(\mathbb{C}^{d-1}))$	$L = a_1, L = a_1^\dagger$

4.2. The Case One $L, H = 0$

Let L be as in (4.1). If $H = 0$, then $\delta_H = 0$. In particular

$$\mathcal{M} = \text{Lin}_{\mathbb{R}}\{v - u, u - v, i(v + u), i(u + v)\} = \text{Lin}_{\mathbb{R}}\{v - u, i(v + u)\}$$

and, since both v and u cannot be zero in our framework, $\dim_{\mathbb{R}} \mathcal{M}$ is either 1 or 2. If it is equal to 1 (first case), clearly $\mathcal{M} \cap \mathcal{M}' = \mathcal{M}$ and $\mathcal{M}^r = \{0\}$ therefore $d_c = 1$ and $d_r = 0$. It follows that $d_f = d - 1$ and $\mathcal{N}(\mathcal{T})$ is a von Neumann algebra unitarily equivalent to $L^\infty(\mathbb{R}; \mathbb{C}) \overline{\otimes} \mathcal{B}(\Gamma(\mathbb{C}^{d-1}))$. If $\dim_{\mathbb{R}} \mathcal{M} = 2$ then, since \mathcal{M}^r is a symplectic space, its real dimension must be even and so we distinguish two cases: $d_r = 0, d_c = 2$ (second case) and $d_r = 1, d_c = 0$ (third case). If $d_c = 2$, again $\mathcal{M} \cap \mathcal{M}' = \mathcal{M}$, and $\mathcal{N}(\mathcal{T})$ is a von Neumann algebra $L^\infty(\mathbb{R}^2; \mathbb{C}) \overline{\otimes} \mathcal{B}(\Gamma(\mathbb{C}^{d-2}))$. If $d_c = 0, d_r = 1$, then $d_f = d - 1$ and $\mathcal{N}(\mathcal{T})$ is a von Neumann algebra $\mathcal{B}(\Gamma(\mathbb{C}^{d-1}))$. This classification is summarized by Table 1 in which the last column labeled “ L ” contains possible choices of the operator L that realize each case.

In the last part of the section we will characterize each case by just looking directly at the operator L instead of computing \mathcal{M} .

Suppose L is self-adjoint. In this case \mathcal{V} is composed of only one vector which is of the form $[\bar{v}, v]^T$. Therefore $\mathcal{M} = \text{Lin}_{\mathbb{R}}\{iv\}$ and $d_c = 1$, while $d_r = 0$ (1st case). Consider now instead the case L normal but not self-adjoint. An explicit computation shows that $0 = [L, L^*] = \|v\|^2 - \|u\|^2$ on D . This condition shows that $\mathcal{M} = \mathcal{M} \cap \mathcal{M}'$ since

$$\Im\langle v - u, i(v + u) \rangle = \|v\|^2 - \|u\|^2 = 0.$$

Moreover $u \neq v$ since L is not self-adjoint, hence $d_c = 2$ (2nd case). If L is not even normal (i.e. $\|v\|^2 \neq \|u\|^2$) then by the previous calculations $d_c = 0$ and $d_r = 1$ (3rd case).

Summing up: the 1st case arises when L is self-adjoint, the case 2nd case arises when L is normal but not self-adjoint and the 3rd case arises when L is not normal or, equivalently $\|v\|^2 \neq \|u\|^2$.

In the last case it can be shown that when $\|v\|^2 > \|u\|^2$ (resp. $\|v\|^2 < \|u\|^2$) there exists a Bogoliubov transformation changing L to a multiple of the annihilation operator a_1 (resp. creation operator a_1^\dagger).

4.3. Two Bosons in a Common Bath

The following model for the open quantum system of two bosons in a common environment has been considered in Ref. [8]. Here $d = 2$ and H is as in equation 2.6

with $\kappa = \zeta = 0$. The completely positive part of the GKLS generator \mathcal{L} is

$$\frac{1}{2} \sum_{j,k=1,2} \gamma_{jk}^- a_j^\dagger X a_k + \frac{1}{2} \sum_{j,k=1,2} \gamma_{jk}^+ a_j X a_k^\dagger \tag{4.2}$$

where $(\gamma_{jk}^\pm)_{j,k=1,2}$ are positive definite 2×2 matrices.

Note that, by a change of phase $a_1 \rightarrow e^{i\theta_1} a_1$, $a_1^\dagger \rightarrow e^{-i\theta_1} a_1^\dagger$, $a_2 \rightarrow e^{i\theta_2} a_2$, $a_2^\dagger \rightarrow e^{-i\theta_2} a_2^\dagger$, we can always assume that $(\gamma_{jk}^-)_{j,k=1,2}$ is *real* symmetric. Write the spectral decomposition

$$\gamma^\pm = \lambda_\pm |\varphi^\pm\rangle\langle\varphi^\pm| + \mu_\pm |\psi^\pm\rangle\langle\psi^\pm|$$

where the vectors φ^-, ψ^- have *real* components. Rewrite the first term of (4.2) as

$$\begin{aligned} \sum_{j,k=1,2} \gamma_{jk}^- a_j^\dagger X a_k &= \lambda_- \sum_{j,k=1,2} \varphi_j^- \varphi_k^- a_j^\dagger X a_k + \mu_- \sum_{j,k=1,2} \psi_j^- \psi_k^- a_j^\dagger X a_k \\ &= \lambda_- \left(\sum_{j=1,2} \varphi_j^- a_j^\dagger \right) X \left(\sum_{k=1,2} \varphi_k^- a_k \right) \\ &\quad + \mu_- \left(\sum_{j=1,2} \psi_j^- a_j^\dagger \right) X \left(\sum_{k=1,2} \psi_k^- a_k \right) \end{aligned}$$

and write in a similar way the second term of (4.2)

$$\begin{aligned} \sum_{j,k=1,2} \gamma_{jk}^+ a_j X a_k^\dagger &= \lambda_+ \left(\sum_{j=1,2} \overline{\varphi_j^+} a_j \right) X \left(\sum_{k=1,2} \varphi_k^+ a_k^\dagger \right) \\ &\quad + \mu_+ \left(\sum_{j=1,2} \overline{\psi_j^+} a_j \right) X \left(\sum_{k=1,2} \psi_k^+ a_k^\dagger \right) \end{aligned}$$

We can represent \mathcal{L} in a generalized GKLS form with a number of Kraus operators L_ℓ depending on the number of strictly positive eigenvalues among λ_\pm, μ_\pm .

$$\begin{aligned} L_1 &= \lambda_-^{1/2} \sum_{k=1,2} \varphi_k^- a_k & L_2 &= \mu_-^{1/2} \sum_{k=1,2} \psi_k^- a_k \\ L_3 &= \lambda_+^{1/2} \sum_{k=1,2} \overline{\varphi_k^+} a_k^\dagger & L_4 &= \mu_+^{1/2} \sum_{k=1,2} \overline{\psi_k^+} a_k^\dagger \end{aligned}$$

Relabelling if necessary, we can always assume $0 \leq \lambda_- \leq \mu_-$ and $0 \leq \lambda_+ \leq \mu_+$.

We begin our analysis by considering the case where $H = 0$.

If $\lambda_- > 0$ (or $\lambda_+ > 0$) then there are four vectors v, u in the defining set of \mathcal{M} namely

$$\mathcal{M} = \text{Lin}_{\mathbb{R}} \{ \varphi^-, \psi^-, i\varphi^-, i\psi^- \} \quad (\text{or } = \text{Lin}_{\mathbb{R}} \{ \varphi^+, \psi^+, i\varphi^+, i\psi^+ \})$$

thus $\mathcal{M}' = \{0\}$ and $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbf{1}$.

Suppose now that $\lambda_+ = \lambda_- = 0$ and $\mu_-, \mu_+ > 0$ so that there are only two Kraus operators, the above L_2 and L_4 and

$$\mathcal{M} = \text{Lin}_{\mathbb{R}} \{ \psi^-, \psi^+, i\psi^-, i\psi^+ \}.$$

It follows that, if ψ^-, ψ^+ are \mathbb{R} -linearly independent, we have again $\mathcal{M} = \mathbb{C}^2$ whence $\mathcal{M}' = \{0\}$ and $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbf{1}$. Otherwise, if ψ^+ is a *real* non-zero multiple of ψ^- , then, as ψ^\pm and $i\psi^\pm$ are \mathbb{R} -linearly independent, the real dimension of \mathcal{M} and \mathcal{M}' is two, $\mathcal{M} \cap \mathcal{M}' = \{0\}$ so that $\mathcal{N}(\mathcal{T})$ is isomorphic to $\mathcal{B}(\Gamma(\mathbb{C}))$.

It is not difficult to see that, in any case, the dimension of \mathcal{M} cannot be 1 or 3 (because creation and annihilation operator always appear separately in different Kraus operators L , never in the same).

Summarizing: $\mathcal{N}(\mathcal{T})$ is non-trivial and isomorphic to $\mathcal{B}(\Gamma(\mathbb{C}))$ if and only if γ^+ and γ^- are rank-one and commute.

Finally, if we consider a non-zero H , it is clear that $\mathcal{N}(\mathcal{T})$ is always trivial unless γ^+ and γ^- are rank-one, commute and their one-dimensional range is an eigenvector for Ω and Ω^T .

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Appendix A: Construction of Gaussian QMSs From the GKLS Generator

In this section we outline how one can construct the minimal quantum dynamical semigroup associated with operators H, L_ℓ and following [16], Section 3.3. The first step is to prove that the closure of the operator G

$$G = -\frac{1}{2} \sum_{\ell=1}^m L_\ell^* L_\ell - iH \quad (4.3)$$

defined on the domain D generates a strongly continuous contraction semigroup. To this end we recall the result due to Palle E.T. Jorgensen (see [23], Theorem 2).

Theorem 4.1. *Let G be a dissipative linear operator on a Hilbert space \mathfrak{h} . Let $(D_n)_{n \geq 1}$ be an increasing family of closed subspaces of \mathfrak{h} whose union is dense in \mathfrak{h} and contained in the domain of G and let P_{D_n} be the orthogonal projection of \mathfrak{h} onto D_n . Suppose that there exists an integer n_0 such that $GD_n \subset D_{n+n_0}$ for all $n \geq 1$. Then the closure \overline{G} generates a strongly continuous contraction semigroup on \mathfrak{h} and $\cup_{n \geq 1} D_n$ is a core for \overline{G} , if there exists a sequence $(c_n)_{n \geq 1}$ in \mathbb{R}_+ such that $\|GP_{D_n} - P_{D_n}GP_{D_n}\| \leq c_n$ for all n and*

$$\sum_{n=1}^{\infty} c_n^{-1} = \infty$$

We are now able to prove the following proposition.

Proposition 4.2. *The operator \overline{G} is the infinitesimal generator of a strongly continuous contraction semigroup on \mathfrak{h} and D is a core for this operator.*

Proof. We apply Theorem 4.1 with D_n the linear manifold spanned by vectors $e(n_1, \dots, n_d)$ with $n_1 + \dots + n_d \leq n$. Clearly $D = \cup_{n \geq 1} D_n$. The operator G is obviously densely defined and dissipative. Therefore it is closable (see e.g. [5], Lemma 3.1.14) and its closure, denoted \overline{G} is dissipative. Clearly, by the explicit form of the action of creation and annihilation operators on vectors $e(n_1, \dots, n_d)$, the operator G maps D_n into D_{n+2} for all $n \geq 0$.

A straightforward computation using (4.3) yields

$$(GP_{D_n} - P_{D_n}GP_{D_n}) = - \left(\frac{1}{2} \sum_{k,j=1}^d \left(\sum_{\ell=1}^m v_{\ell k} u_{\ell j} + i\kappa \right) a_k^\dagger a_j^\dagger + \frac{i}{2} \sum_{k=1}^d \zeta_k a_k^\dagger \right) P_{D_n},$$

namely the non-zero part is the one involving only creations. Let us fix $u = \sum_{|\alpha| \leq n} r_\alpha e_\alpha$ a vector in D_n , where $\alpha = (\alpha(1), \dots, \alpha(d))$ is a multi-index, $|\alpha| = \alpha(1) + \dots + \alpha(d)$, and the vector $e_\alpha^T = (e_{\alpha(1)}, \dots, e_{\alpha(d)})$. Clearly $a_j^\dagger u \in D_{n+1}$ and

$$\|a_j^\dagger u\|^2 \leq \sum_{|\alpha| \leq n} |r_\alpha|^2 (\alpha(j) + 1) \|e_{\alpha+1_j}\|^2 \leq (n + 1) \|u\|^2.$$

Therefore we have also

$$\|a_j^\dagger a_k^\dagger u\|^2 \leq (n + 2) \|a_k^\dagger u\|^2 \leq (n + 2)(n + 1) \|u\|^2 \leq (n + 2)^2 \|u\|^2.$$

This means that

$$\begin{aligned} \|(GP_{D_n} - P_{D_n}GP_{D_n})u\| &\leq \frac{1}{2} \sum_{j,k=1}^d \left\| \left[\sum_{\ell} (\overline{v}_{\ell \bullet})^* u_{\ell} + i\kappa \right]_{jk} a_j^\dagger a_k^\dagger u \right\| \\ &\quad + \sum_{j=1}^d \| \zeta_j a_j^\dagger u \| \\ &\leq c(n + 2) \|u\| / 2 \end{aligned}$$

with $c > 0$ a constant that does not depend on n . Eventually, since the series $\sum_{n \geq 1} (n + 2)^{-1}$ diverges, we can apply Theorem 4.1 and the proposition is proved. \square

A similar argument allows us to prove the following

Proposition 4.3. *The closure Φ of the operator $\sum_{1 \leq \ell \leq m} L_\ell^* L_\ell$ defined on the domain D is essentially self-adjoint.*

Let us denote by G also the closure of the operator G to simplify the notation. Using standard arguments, the operators L_ℓ can be extended to the domain of G and further extended to the domain

$$\left\{ u \in \mathfrak{h} \mid u = \sum_{\alpha} r_{\alpha} e_{\alpha}, \sum_{\alpha} |\alpha| |r_{\alpha}|^2 < \infty \right\},$$

where $\alpha = (\alpha(1), \dots, \alpha(d))$ is a multi-index, $|\alpha| = \alpha(1) + \dots + \alpha(d)$, and the vector $e_{\alpha} = (e_{\alpha(1)}, \dots, e_{\alpha(d)})$.

In the next section we will show that the minimal semigroup is identity preserving and so it is a well defined QMS, whose predual semigroup is trace preserving.

4.4. Conservativity

We will establish conservativity by applying the Chebotarev-Fagnola sufficient condition (see [9] , [16] section 3.5). More precisely, we will apply the following result:

Theorem 4.4. *Suppose that:*

1. *the operator G is the infinitesimal generator of a strongly continuous contraction semigroup $(P_t)_{t \geq 0}$ in \mathfrak{h} ,*
2. *the domain of the operators $(L_\ell)_{\ell \geq 1}$ contains the domain of G and, for every $u \in D(G)$, we have*

$$\langle u, Gu \rangle + \langle Gu, u \rangle + \sum_{\ell} \langle L_\ell u, L_\ell u \rangle = 0,$$

3. *there exists a self-adjoint operator C with domain coinciding with the domain of G and a core D for C with the following properties*
 - (a) *$L_\ell(D) \subset D(C^{1/2})$ for all $\ell \geq 1$,*
 - (b) *there exists a self-adjoint operator Φ such that*

$$-2\Re \langle u, Gu \rangle = \langle u, \Phi u \rangle \leq \langle u, Cu \rangle$$

for all $u \in D$,

- (c) *there exists a positive constant b such that the inequality*

$$2\Re \langle Cu, Gu \rangle + \sum_{\ell \geq 1} \langle C^{1/2} L_\ell u, C^{1/2} L_\ell u \rangle \leq b \langle u, Cu \rangle$$

holds for every $u \in D$.

Then the minimal quantum dynamical semigroup is Markov.

In order to check the above conditions one should proceed with computations on quadratic forms. However, these are equivalent to algebraic computations of the action of the formal generator \mathcal{L} on first and second order polynomials is a_j, a_j^\dagger

therefore we will go on with algebraic computations so as to reduce the clutter of the notation.

Lemma 4.5. *It holds*

$$\begin{aligned} 2\mathcal{L}(a_k) &= a^\dagger \left((U^T V - V^T U - 2i\kappa)_{k\bullet} \right) + a \left((U^* U - V^* V + 2i\bar{\Omega})_{k\bullet} \right) - i\zeta_k \mathbb{1}, \\ 2\mathcal{L}(a_k^\dagger) &= a \left((U^T V - V^T U - 2i\kappa)_{k\bullet} \right) + a^\dagger \left((U^* U - V^* V + 2i\bar{\Omega})_{k\bullet} \right) + i\bar{\zeta}_k \mathbb{1}. \end{aligned}$$

Proof. First write

$$\mathcal{L}_0(X) := \mathcal{L}(X) - i[H, X] = \frac{1}{2} \sum_{\ell=1}^m (L_\ell^* [X, L_\ell] + [L_\ell^*, X] L_\ell).$$

By the CCR one has

$$\begin{aligned} [a_k, L_\ell] &= \sum_{j=1}^d [a_k, u_{\ell j} a_j^\dagger] = u_{\ell k} \mathbb{1}, \\ [L_\ell^*, a_k] &= [a_k^\dagger, L_\ell]^* = \sum_{j=1}^d [a_k^\dagger, \bar{v}_{\ell j} a_j]^* = -v_{\ell k} \mathbb{1} \end{aligned}$$

Therefore we obtain that

$$\mathcal{L}_0(a_k) = \frac{1}{2} \sum_{\ell=1}^m \left\{ \left(\sum_{j=1}^d v_{\ell j} a_j^\dagger + \bar{u}_{\ell j} a_j \right) u_{\ell k} - v_{\ell k} \left(\sum_{j=1}^d \bar{v}_{\ell j} a_j + u_{\ell j} a_j^\dagger \right) \right\}.$$

and

$$[H, a_k] = -\frac{i\zeta_k}{2} \mathbb{1} - \sum_{j=1}^d \left(\Omega_{kj} a_j + \kappa_{kj} a_j^\dagger \right).$$

which both lead to

$$\begin{aligned} \mathcal{L}(a_k) &= i[H, a_k] + \mathcal{L}_0(a_k) \\ &= -\frac{\zeta_k}{2} \mathbb{1} - i \sum_{j=1}^d \left(\Omega_{kj} a_j + \kappa_{kj} a_j^\dagger \right) \\ &\quad + \frac{1}{2} \sum_{\ell=1}^m \left\{ \left(\sum_{j=1}^d v_{\ell j} a_j^\dagger + \bar{u}_{\ell j} a_j \right) u_{\ell k} - v_{\ell k} \left(\sum_{j=1}^d \bar{v}_{\ell j} a_j + u_{\ell j} a_j^\dagger \right) \right\} \end{aligned}$$

Using the last equality and $\mathcal{L}(a_k^\dagger) = \mathcal{L}(a_k)^*$ concludes the proof. □

The following formula is verified for any generator \mathcal{L} of a QMS:

Lemma 4.6. *For all $X, Y \in \mathcal{B}(\mathfrak{h})$*

$$\mathcal{L}(XY) = X\mathcal{L}(Y) + \mathcal{L}(X)Y + \sum_{\ell=1}^m [L_\ell, X^*]^* [L_\ell, Y]. \tag{4.4}$$

Proof. Let $\mathcal{L}_0(X) = -(1/2) \sum_{\ell=1}^m (L_\ell^* L_\ell X - 2L_\ell^* X L_\ell + X L_\ell^* L_\ell)$ then we can write $\mathcal{L}_0(XY) - X \mathcal{L}_0(Y) - \mathcal{L}_0(X)Y$ as

$$\begin{aligned} & \sum_{\ell=1}^m \left(-\frac{1}{2} L_\ell^* L_\ell XY + L_\ell^* XY L_\ell - \frac{1}{2} XY L_\ell^* L_\ell \right) \\ & + \sum_{\ell=1}^m \left(\frac{1}{2} X L_\ell^* L_\ell Y - X L_\ell^* Y L_\ell + \frac{1}{2} XY L_\ell^* L_\ell \right) \\ & + \sum_{\ell=1}^m \left(\frac{1}{2} L_\ell^* L_\ell XY - L_\ell^* X L_\ell Y + \frac{1}{2} X L_\ell^* L_\ell Y \right) \\ & = \sum_{\ell=1}^m (L_\ell^* XY L_\ell + X L_\ell^* L_\ell Y - X L_\ell^* Y L_\ell - L_\ell^* X L_\ell Y) \\ & = \sum_{\ell=1}^m ([L_\ell^*, X]Y L_\ell - [L_\ell^*, X]L_\ell Y) = \sum_{\ell=1}^m ([L_\ell, X^*]^* [L_\ell, Y]) \end{aligned}$$

Recalling the usual commutator property $[H, XY] = [H, X]Y + X[H, Y]$, we find then

$$\mathcal{L}(XY) - X \mathcal{L}(Y) - \mathcal{L}(X)Y = \mathcal{L}_0(XY) - X \mathcal{L}_0(Y) - \mathcal{L}_0(X)Y.$$

This completes the proof. □

As a final step towards proving conservativity via Theorem 4.4, we prove the following

Proposition 4.7. *Let $C = \sum_{k=1}^d a_k a_k^\dagger$. There exist a constant $b > 0$ such that $\mathcal{L}(C) \leq bC$*

Proof. By Lemmas 4.5 we have that

$$\mathcal{L}(a_k) = \sum_{j=1}^d (w_{kj} a_j^\dagger + z_{kj} a_j) - \frac{i\zeta_k}{2} \mathbb{1}, \quad \mathcal{L}(a_k^\dagger) = \sum_{j=1}^d (\bar{w}_{kj} a_j + \bar{z}_{kj} a_j^\dagger) + \frac{i\bar{\zeta}_k}{2} \mathbb{1}$$

for some complex numbers w_{kj}, z_{kj}, ζ_j . While, by Lemma 4.6, we get

$$\begin{aligned} \mathcal{L}(a_k a_k^\dagger) &= -i\bar{\zeta}_k a_k / 2 + \sum_{j=1}^d (\bar{w}_{kj} a_k a_j + \bar{z}_{kj} a_k a_j^\dagger) \\ &+ i\zeta_k a_k^\dagger / 2 + \sum_{j=1}^d (w_{kj} a_j^\dagger a_k^\dagger + z_{kj} a_j a_k^\dagger) + \|v_{\bullet k}\|^2 \mathbb{1} \\ &= \sum_{j=1}^d (\bar{z}_{kj} a_k a_j^\dagger + z_{kj} a_j a_k^\dagger + \bar{w}_{kj} a_k a_j + w_{kj} a_j^\dagger a_k^\dagger) \\ &+ \frac{i}{2} (\zeta_k a_k^\dagger - \bar{\zeta}_k a_k) + \|v_{\bullet k}\|^2 \mathbb{1} \end{aligned}$$

Note that for each k, j

$$|a_j^\dagger - z_{kj}a_k^\dagger|^2 = a_j a_j^\dagger + |z_{kj}|^2 a_k a_k^\dagger - \bar{z}_{kj} a_k a_j^\dagger - z_{kj} a_j a_k^\dagger \geq 0,$$

it follows that

$$\bar{z}_{kj} a_k a_j^\dagger + z_{kj} a_j a_k^\dagger \leq a_j a_j^\dagger + |z_{kj}|^2 a_k a_k^\dagger \tag{4.5}$$

and, in the same way

$$\bar{w}_{kj} a_k a_j + w_{kj} a_k^\dagger a_j \leq |w_{kj}|^2 a_k a_k^\dagger + a_j a_j^\dagger. \tag{4.6}$$

Finally, from

$$|a_k^\dagger + i\bar{\zeta}_k \mathbb{1}|^2 = a_k a_k^\dagger + |\zeta_k|^2 \mathbb{1} - i\zeta_k a_k^\dagger + i\bar{\zeta}_k a_k \geq 0$$

it follows that

$$i\zeta_k a_k^\dagger - i\bar{\zeta}_k a_k \leq a_k a_k^\dagger + |\zeta_k|^2 \mathbb{1}. \tag{4.7}$$

By (4.5),(4.6), and (4.7)

$$\mathcal{L}(C) \leq \left(3d \max_{1 \leq k \leq d} \left(1 + \sum_{j=1}^d (|z_{kj}|^2 + |w_{kj}|^2) \right) C + \sum_{j=1}^d (|\zeta_j|^2 + \|v_{\bullet j}\|^2) \mathbb{1} \right)$$

since $C \geq d\mathbb{1}$ then $\mathcal{L}(C) \leq bC$ with

$$b = \max \left\{ 3d \max_{1 \leq k \leq d} \left(\sum_{j=1}^d (|z_{kj}|^2 + |w_{kj}|^2) \right), \sum_{j=1}^d (|\zeta_j|^2 + \|v_{\bullet j}\|^2) \right\}.$$

□

We can eventually state the result on conservativity.

Theorem 4.8. *The minimal QDS semigroup generated by the pre-generator (2.5) with H, L_ℓ given by (2.6),(2.7) is Markov.*

Proof. We apply Theorem 4.4 with the operator C given by

$$\text{Dom}(C) = \left\{ u = \sum_{\alpha} u_{\alpha} e(\alpha) \mid \sum_{\alpha} |\alpha|^2 |u_{\alpha}|^2 < \infty \right\}, \quad Cu = \sum_{j=1}^d a_j a_j^\dagger u.$$

Conditions 1 and 2 are satisfied by definition and by Proposition 4.2. For condition 3 one can choose D as in the proof of Proposition 4.2. In this way (a) is easily satisfied, while (c) follows from Proposition 4.7. The operator Φ is the self-adjoint extension of $\sum_{\ell} L_{\ell}^* L_{\ell}$ (defined on the domain D) and is second-order polynomial in a, a^\dagger . Inequalities like (4.6) and (4.5) allows one to show that Φ is majorized by a suitable multiple of C . Replacing C with this suitable multiple the proof is completed by Proposition 4.7. □

4.5. Proof of Theorem 2.4

Following the proof of Theorem 2 in [1] let us start by derivating (2.10) at $t = 0$. Using

$$\frac{d}{dt} W(e^{tZ}z) \Big|_{t=0} = W(z) \sum_{j=1}^d \left((Zz)_j a_j^\dagger - (\overline{Zz})_j a_j + \frac{1}{2} (\overline{z_j} (Zz)_j - (\overline{Zz})_j z_j) \right)$$

(with respect to the norm topology) one has $\mathcal{L}(W(z)) = W(z)Y(z)$, where $Y(z)$ corresponds to the operator

$$\sum_{j=1}^d \left((Zz)_j a_j^\dagger - (\overline{Zz})_j a_j + \frac{1}{2} (\overline{z_j} (Zz)_j - (\overline{Zz})_j z_j) \right) - \frac{1}{2} \Re \langle z, Cz \rangle + i \Re \langle \zeta, z \rangle.$$

Now instead let us rewrite the GKLS pre-generator as

$$\mathcal{L}(W(z)) = i[H, W(z)] + \frac{1}{2} \sum_{\ell=1}^m (L_\ell^* [W(z), L_\ell] + [L_\ell^*, W(z)] L_\ell).$$

Recalling that $\Omega = \Omega^*$, $\kappa = \kappa^T$ and from (2.4) one gets

$$[W(z), L_\ell] = -W(z) (\overline{V}z + U\overline{z})_\ell, \quad [L_\ell^*, W(z)] = W(z) (V\overline{z} + \overline{U}z)_\ell$$

and so

$$\begin{aligned} [H, W(z)] &= W(z) [a(\Omega z + \kappa\overline{z}) + a^\dagger(\Omega z + \kappa\overline{z}) \\ &\quad + \left(\frac{1}{2} \langle z, \Omega z + \kappa\overline{z} \rangle + \frac{1}{2} \overline{\langle z, \Omega z + \kappa\overline{z} \rangle} + \Re \langle \zeta, z \rangle \right) W(z) \end{aligned}$$

At last one finds

$$\begin{aligned} \sum_{\ell=1}^m [L_\ell^*, [W(z), L_\ell]] &= - \sum_{\ell=1}^m W(z) (\overline{V}z + U\overline{z})_\ell (V\overline{z} + \overline{U}z)_\ell \\ &= -W(z) \left(\langle z, \overline{V^*V}z + V^T U\overline{z} \rangle + \overline{\langle z, \overline{U^*U}z + U^T V\overline{z} \rangle} \right). \end{aligned}$$

Using the previous results one finds that $\mathcal{L}(W(z)) = W(z)X(z)$ for some operator $X(z)$ which is explicitly given by

$$\begin{aligned} X(z) &= a^\dagger \left(\left(\frac{\overline{U^*U} - V^*V}{2} + i\Omega \right) z + \left(\frac{U^T V - V^T U}{2} + i\kappa \right) \overline{z} \right) \\ &\quad - a \left(\left(\frac{\overline{U^*U} - V^*V}{2} + i\Omega \right) z + \left(\frac{U^T V - V^T U}{2} + i\kappa \right) \overline{z} \right) \\ &\quad + \frac{1}{2} \langle z, i\Omega z + i\kappa\overline{z} \rangle - \frac{1}{2} \overline{\langle z, i\Omega z + i\kappa\overline{z} \rangle} + i \Re \langle \zeta, z \rangle \\ &\quad - \frac{1}{2} \left(\langle z, \overline{V^*V}z + V^T U\overline{z} \rangle + \overline{\langle z, \overline{U^*U}z + U^T V\overline{z} \rangle} \right). \end{aligned}$$

Since $X(z)$ and $Y(z)$ must coincide for every $z \in \mathbb{C}^d$ the proof of Theorem 2.4 is complete.

Appendix B: Characterization of $\mathcal{N}(\mathcal{T})$

In this section we derive the characterization of $\mathcal{N}(\mathcal{T})$ in terms of iterated commutators. We begin by illustrating the idea of the proof in the case where the operators L_ℓ and H are bounded. For all $x, y \in \mathcal{B}(\mathfrak{h})$, recall the formula (4.4) from Lemma 4.6. Note that, if $x \in \mathcal{N}(\mathcal{T})$ and y is arbitrary then, since $\mathcal{T}_t(y^*x) = \mathcal{T}_t(y^*)\mathcal{T}_t(x)$ by Proposition 3.1 1, taking the derivative at $t = 0$ we get $\mathcal{L}(y^*x) = \mathcal{L}(y^*)x + y^*\mathcal{L}(x)$, therefore

$$\sum_{\ell=1}^m [L_\ell, y]^* [L_\ell, x] = 0. \tag{4.8}$$

If the operators L_ℓ are bounded, we are allowed to take $x = y$, then $[L_\ell, x] = 0$ for all ℓ . Moreover, since x^* also belongs to $\mathcal{N}(\mathcal{T})$, taking the adjoint of $[L_\ell, x^*] = 0$, x also commutes with all the operators L_ℓ^* and $\mathcal{L}(x) = i[H, x]$. Clearly, since $\mathcal{N}(\mathcal{T})$ is \mathcal{T}_t -invariant, $\mathcal{L}(x) = \lim_{t \rightarrow 0} (\mathcal{T}_t(x) - x)/t$ belongs to $\mathcal{N}(\mathcal{T})$. Therefore $[L_\ell, [H, x]] = 0$ for all ℓ and, by the Jacobi identity

$$[x, [H, L_\ell]] = -[H, [L_\ell, x]] - [L_\ell, [x, H]] = 0.$$

In this way, one can show inductively that x commutes with the iterated commutators (3.2).

If the operators L_ℓ, H are unbounded, one has to cope with several problems. The operator \mathcal{L} is unbounded and, even if we choose x, y in the domain of \mathcal{L} , it is not clear whether y^*x belongs to the domain of \mathcal{L} (see [17]). Multiplication of generalized commutators $[L_\ell, y] [L_\ell, x]$ may not be defined. If we choose a “nice” $y \in \text{Dom}(\mathcal{L})$ then it is not clear whether we can take $x = y$ because we do not know a priori if our “nice” y belongs to $\mathcal{N}(\mathcal{T})$.

We begin the analysis of $\mathcal{N}(\mathcal{T})$ by a few preliminary lemmas.

Lemma 4.9. *The following derivative exists with respect norm topology for all $z \in \mathbb{C}$*

$$\frac{d}{dt} \mathcal{T}_t(W(z))e_g \Big|_{t=0} = G^*W(z)e_g + \sum_{\ell=1}^m L_\ell^*W(z)L_\ell e_g + W(z)Ge_g$$

Proof. The right-hand side operator $G^*W(z) + \sum_{\ell=1}^m L_\ell^*W(z)L_\ell + W(z)G$ is unbounded (for $z \neq 0$) therefore $W(z)$ does not belong to the domain of \mathcal{L} but we can consider the quadratic form $\mathcal{L}(W(z))$ on $D \times D$. Differentiability of functions $t \mapsto \langle \xi', \mathcal{T}_t(x)\xi \rangle$ also holds for ξ, ξ' in the linear span of exponential vectors. Therefore, for all such ξ , we have (Theorem 2.3)

$$\langle \xi, (\mathcal{T}_t(W(z)) - W(z) - t\mathcal{L}(W(z)))e_g \rangle = \int_0^t \langle \xi, \mathcal{L}(\mathcal{T}_s(W(z)) - W(z))e_g \rangle ds.$$

Recalling that $\mathcal{T}_s(W(z)) = \varphi_z(s)W(e^{sZ}z)$ as in (2.10) for a complex valued function φ such that $\lim_{s \rightarrow 0} \varphi_z(s) = 1$, the right-hand side integrand can be written as

$$(\varphi_z(s) - 1) \langle \xi, \mathcal{L}(W(e^{sZ}z))e_g \rangle + \langle \xi, \mathcal{L}(W(e^{sZ}z) - W(z))e_g \rangle$$

A long but straightforward computation shows the function

$$s \mapsto \mathcal{L}(W(e^{sZ}z) - W(z))e_g$$

is continuous vanishing at $s = 0$ and the function $s \mapsto \mathcal{L}(W(e^{sZ}z))e_g$ is bounded with respect to the Fock space norm. Therefore, taking suprema for $\xi \in \Gamma(\mathbb{C}^d)$, $\|\xi\| = 1$, we find the inequalities

$$\begin{aligned} \|(\mathcal{T}_t(W(z)) - W(z) - t\mathcal{L}(W(z)))e_g\| &\leq \int_0^t |\varphi_z(s) - 1| \|\mathcal{L}(W(e^{sZ}z))e_g\| ds \\ &\quad + \int_0^t \|\mathcal{L}(W(e^{sZ}z) - W(z))e_g\| ds \end{aligned}$$

The conclusion follows dividing by t and taking the limit as $t \rightarrow 0^+$. □

Lemma 4.10. *Let $x \in \mathcal{N}(\mathcal{T})$. For all exponential vectors e_g, e_f and all Weyl operators $W(z)$ we have*

$$\sum_{\ell=1}^m \langle [L_\ell, W(-z)] e_g, x L_\ell e_f \rangle = \sum_{\ell=1}^m \langle L_\ell^* [L_\ell, W(-z)] e_g, x e_f \rangle. \tag{4.9}$$

Proof. If $x \in \mathcal{N}(\mathcal{T})$, then, for all $g, f, z \in \mathbb{C}^d$ and $t \geq 0$ we have

$$\begin{aligned} \langle e_g, (\mathcal{T}_t(W(z)x) - W(z)x)e_f \rangle &= \langle e_g, (\mathcal{T}_t(W(z))\mathcal{T}_t(x) - W(z)x)e_f \rangle \\ &= \langle e_g, (\mathcal{T}_t(W(z)) - W(z))x e_f \rangle + \langle e_g, W(z)(\mathcal{T}_t(x) - x)e_f \rangle \\ &\quad + \langle (\mathcal{T}_t(W(-z)) - W(-z))e_g, (\mathcal{T}_t(x) - x)e_f \rangle \end{aligned}$$

By Lemma 4.9 means the norm limit

$$\lim_{t \rightarrow 0^+} \frac{(\mathcal{T}_t(W(-z)) - W(-z))e_g}{t}$$

exists, therefore $\sup_{t > 0} t^{-1} \|\mathcal{T}_t(W(-z)) - W(-z)\|e_g < +\infty$. Moreover

$$\begin{aligned} \|(\mathcal{T}_t(x) - x)e_f\|^2 &= \langle e_f, (\mathcal{T}_t(x^*) - x^*)(\mathcal{T}_t(x) - x)e_f \rangle \\ &\leq \langle e_f, (\mathcal{T}_t(x^*x) - x^*\mathcal{T}_t(x) - \mathcal{T}_t(x^*)x + x^*x)e_f \rangle \end{aligned}$$

which tends to 0 as $t \rightarrow 0^+$ by weak* continuity of \mathcal{T}_t . As a result

$$\lim_{t \rightarrow 0^+} t^{-1} \langle e_g, (\mathcal{T}_t(W(z)) - W(z))(\mathcal{T}_t(x) - x)e_f \rangle = 0,$$

therefore

$$\begin{aligned} \lim_{t \rightarrow 0^+} t^{-1} \langle e_g, (\mathcal{T}_t(W(z)x) - W(z)x)e_f \rangle &= \lim_{t \rightarrow 0^+} t^{-1} \langle e_g, W(z)(\mathcal{T}_t(x) - x)e_f \rangle \\ &\quad + \lim_{t \rightarrow 0^+} t^{-1} \langle (\mathcal{T}_t(W(-z)) - W(-z))e_g, x e_f \rangle \end{aligned}$$

and we get

$$\begin{aligned} &\langle Ge_g, W(z)x e_f \rangle + \sum_{\ell=1}^m \langle L_\ell e_g, W(z)x L_\ell e_f \rangle + \langle e_g, W(z)x Ge_f \rangle \\ &= \left\langle \left(G^*W(-z) + \sum_{\ell} L_\ell^*W(-z)L_\ell + W(-z)G \right) e_g, x e_f \right\rangle \\ &\quad + \langle GW(-z)e_g, x e_f \rangle + \sum_{\ell=1}^m \langle L_\ell W(-z)e_g, x L_\ell e_f \rangle + \langle W(-z)e_g, x Ge_f \rangle. \end{aligned}$$

The first term in the left-hand side cancels with the third term in right-hand side and last terms in both sides cancel as well. Noting that

$$G^*W(-z)e_g + GW(-z)e_g = - \sum_{\ell} L_{\ell}^*L_{\ell}W(-z)e_g$$

adding the first and fourth terms in the right-hand side, we find

$$\begin{aligned} \sum_{\ell=1}^m \langle L_{\ell}e_g, W(z)xL_{\ell}e_f \rangle &= \sum_{\ell=1}^m [\langle L_{\ell}^*W(-z)L_{\ell}e_g, xe_f \rangle + \langle L_{\ell}W(-z)e_g, xL_{\ell}e_f \rangle] \\ &\quad - \sum_{\ell=1}^m \langle L_{\ell}^*L_{\ell}W(-z)e_g, xe_f \rangle. \end{aligned}$$

Rearranging terms we get (4.9) which is a weak form of identity (4.8). □

The following lemma serves to get (4.9) for each ℓ fixed without summation, taking advantage of the arbitrariness of z .

Lemma 4.11. *For all $\ell_{\bullet} \in \{1, 2, \dots, d\}$ fixed there exists $z \in \mathbb{C}^d$ such that*

$$\sum_{j=1}^d (\bar{v}_{ij}z_j + u_{ij}\bar{z}_j) = \delta_{i,\ell_{\bullet}} = \begin{cases} 1 & \text{if } i = \ell_{\bullet} \\ 0 & \text{if } i \neq \ell_{\bullet} \end{cases}$$

Proof. Note that $\bar{V}z + U\bar{z}$ arises from the map composition

$$\begin{array}{ccc} J_c & [\bar{V} | U] \\ z \rightarrow \begin{bmatrix} z \\ \bar{z} \end{bmatrix} & \longrightarrow \bar{V}z + U\bar{z} \end{array}$$

Let $(\phi_{\ell})_{1 \leq \ell \leq m}$ be an orthonormal basis of \mathbb{C}^m . We look for a $z \in \mathbb{C}^m$ solving the real linear system

$$[\bar{V} | U] J_c z = \phi_{\ell_{\bullet}}$$

Since

$$\text{Ran}([\bar{V} | U] J_c) = \ker([\bar{V} | U] J_c)^{\perp} = \ker\left(J_c^T [\bar{V} | U]^T\right)^{\perp},$$

J_c is one-to-one and, by the minimality assumption (2.8)

$$\ker([\bar{V} | U]^T) = \ker\left(\begin{bmatrix} V^* \\ U^T \end{bmatrix}\right) = \ker(V^*) \cap \ker(U^T) = \{0\},$$

we find $\text{Ran}([\bar{V} | U] J_c) = \mathbb{C}^m$ and the proof is complete. □

Proposition 4.12. *The decoherence-free subalgebra $x \in \mathcal{N}(\mathcal{T})$ is contained in the generalized commutant of the operators L_{ℓ}, L_{ℓ}^* $1 \leq \ell \leq m$.*

Proof. For a Weyl operator $W(z)$ we have

$$[L_{\ell}, W(z)] = \sum_{j=1}^d [\bar{v}_{\ell j}a_j + u_{\ell j}a_j^{\dagger}, W(z)] = \sum_{j=1}^d (\bar{v}_{\ell j}z_j + u_{\ell j}\bar{z}_j) W(z).$$

and (4.9) becomes

$$\begin{aligned} & \sum_{\ell=1}^m \sum_{j=1}^d (\bar{v}_{\ell j} z_j + u_{\ell j} \bar{z}_j) \langle W(-z)e_g, xL_{\ell}e_f \rangle \\ &= \sum_{\ell=1}^m \sum_{j=1}^d (\bar{v}_{\ell j} z_j + u_{\ell j} \bar{z}_j) \langle L_{\ell}^*W(-z)e_g, xe_f \rangle. \end{aligned}$$

By Lemma 4.11, choosing some special $z_{\ell} \in \mathbb{C}^d$, we find

$$\langle W(-z_{\ell})e_g, xL_{\ell}e_f \rangle = \langle L_{\ell}^*W(-z_{\ell})e_g, xe_f \rangle$$

for all $g, f \in \mathbb{C}^d$ and all ℓ . Therefore, by the arbitrariness of g and the explicit action of Weyl operators on exponential vectors

$$\langle e_w, xL_{\ell}e_f \rangle = \langle L_{\ell}^*e_w, xe_f \rangle$$

for all $w, f \in \mathbb{C}^d$ and all ℓ . Since exponential vectors form a core for L_{ℓ}^* and L_{ℓ} is closed, this implies that xe_f belongs to the domain of L_{ℓ} and $L_{\ell}xe_f = xL_{\ell}e_f$, namely $xL_{\ell} \subseteq L_{\ell}x$.

Replacing x with x^* we find $x^*L_{\ell} \subseteq L_{\ell}x^*$ and standard results on the adjoint of products of operators (see e.g. [24] 5.26 p. 168) lead to the inclusions

$$xL_{\ell}^* \subseteq (L_{\ell}x^*)^* \subseteq (x^*L_{\ell})^* = L_{\ell}x^*.$$

It follows that x belongs to the generalised commutant of the set of Kraus operators, namely $\{L_{\ell}, L_{\ell}^* \mid 1 \leq \ell \leq m\}$. □

Lemma 4.13. *The domain $\text{Dom}(N^{n/2})$ is e^{itH} -invariant for all $t \in \mathbb{R}$ and there exists a constant $c_n > 0$ such that*

$$\left\| (N + \mathbb{1})^{n/2} e^{itH} \xi \right\|^2 \leq e^{c_n |t|} \left\| (N + \mathbb{1})^{n/2} \xi \right\|^2 \tag{4.10}$$

for all $\xi \in \text{Dom}(N^{n/2})$.

Proof. Consider Yosida approximations of the identity operator $(\mathbb{1} + \epsilon(N + \mathbb{1}))^{-1}$ for all $\epsilon > 0$ and bounded approximations $X_{\epsilon} = (N + \mathbb{1})^n (\mathbb{1} + \epsilon(N + \mathbb{1}))^{-n}$ of the n -the power of $N + \mathbb{1}$. Note that, the domain D is invariant for these operators and also H invariant. For all $u \in D$, setting $v_{\epsilon} = (\mathbb{1} + \epsilon N)^{-n} u$ we have

$$\langle u, (X_{\epsilon}H - HX_{\epsilon})u \rangle = \langle v_{\epsilon}, ((N + \mathbb{1})^n H(\mathbb{1} + \epsilon N)^n - (\mathbb{1} + \epsilon N)^n H(N + \mathbb{1})^n)v_{\epsilon} \rangle$$

Compute

$$\begin{aligned} & ((N + \mathbb{1})^n H(\mathbb{1} + \epsilon(N + \mathbb{1}))^n - (\mathbb{1} + \epsilon(N + \mathbb{1}))^n H(N + \mathbb{1})^n) \\ &= \sum_{k=0}^n \binom{n}{k} \epsilon^k ((N + \mathbb{1})^n H(N + \mathbb{1})^k - (N + \mathbb{1})^k H(N + \mathbb{1})^n) \\ &= \sum_{k=0}^n \binom{n}{k} \epsilon^k (N + \mathbb{1})^k [(N + \mathbb{1})^{n-k}, H](N + \mathbb{1})^k \end{aligned}$$

and noting that the commutator $[(N + \mathbb{1})^{n-k}, H]$ is a polynomial in a_j, a_k^\dagger of order $2(n - k)$. This implies that we can find a constant c_n such that $|\langle u', [(N + \mathbb{1})^{n-k}, H]u' \rangle| \leq c_n \|(N + \mathbb{1})^{(n-k)/2}u'\|^2$ (for $u' \in D$) and so we get the inequality

$$\begin{aligned} |\langle u, X_\epsilon H u \rangle - \langle H u, X_\epsilon u \rangle| &\leq c_n \sum_{k=0}^n \binom{n}{k} \epsilon^k \langle v_\epsilon, (N + \mathbb{1})^{n+k} v_\epsilon \rangle \\ &= c_n \langle v_\epsilon, (N + \mathbb{1})^n (\mathbb{1} + \epsilon(N + \mathbb{1}))^n v_\epsilon \rangle \\ &= c_n \langle u, X_\epsilon u \rangle. \end{aligned}$$

The above inequality extends to $u \in \text{Dom}(H)$ by density.

Now, for all $u \in \text{Dom}(H)$ and $t \geq 0$, we have

$$\begin{aligned} \left\| X_\epsilon^{1/2} e^{itH} u \right\|^2 - \left\| X_\epsilon^{1/2} u \right\|^2 &= \int_0^t \frac{d}{ds} \left\| X_\epsilon^{1/2} e^{isH} u \right\|^2 ds \\ &= i \int_0^t (\langle e^{isH} u, X_\epsilon H e^{isH} u \rangle - \langle H e^{isH} u, X_\epsilon e^{isH} u \rangle) ds \\ &\leq c_n \int_0^t \left\| X_\epsilon^{1/2} e^{isH} u \right\|^2 ds. \end{aligned}$$

Gronwall’s lemma implies and a similar argument for $t < 0$ yield

$$\left\| X_\epsilon^{1/2} e^{itH} u \right\|^2 \leq e^{c_n |t|} \left\| X_\epsilon^{1/2} u \right\|^2$$

for all $t \in \mathbb{R}$. Considering $u \in D$ and taking the limit as ϵ goes to zero we get (4.10) for $\xi \in D$ and, finally for $\xi \in \text{Dom}(N^{n/2})$ because D is a core for $N^{n/2}$. \square

Lemma 4.14. *For all j there exists $M_a(\mathbb{C})$ valued analytic functions H^-, H^+ such that*

$$e^{-itH} a_j e^{itH} \xi = \sum_{k=1}^d \left(H_{jk}^-(t) e^{-itH} a_k e^{itH} \xi + H_{jk}^+(t) e^{-itH} a_k^\dagger e^{itH} \xi \right)$$

for all $t \in \mathbb{R}, \xi \in \text{Dom}(N)$.

Proof. For all $\xi', \xi \in \text{Dom}(N)$ we have

$$\begin{aligned} \frac{d}{dt} \langle \xi', e^{-itH} a_j e^{itH} \xi \rangle &= \lim_{s \rightarrow 0} \frac{1}{s} \left\langle \xi', \left(e^{-i(t+s)H} a_j e^{i(t+s)H} - e^{-itH} a_j e^{itH} \right) \xi \right\rangle \\ &= \lim_{s \rightarrow 0} s^{-1} \left\langle \left(e^{i(t+s)H} - e^{itH} \right) \xi', a_j e^{itH} \xi \right\rangle \\ &\quad + \lim_{s \rightarrow 0} s^{-1} \left\langle a_j^\dagger e^{itH} \xi', \left(e^{i(t+s)H} - e^{itH} \right) \xi \right\rangle \\ &= \langle iH e^{itH} \xi', a_j e^{itH} \xi \rangle + \langle a_j^\dagger e^{itH} \xi', iH e^{itH} \xi \rangle. \end{aligned}$$

Now, for all $u, v \in D$ we have

$$\langle iH v, a_j u \rangle + \langle a_j^\dagger v, iH u \rangle = -i \langle v, [H, a_j] u \rangle = \sum_{k=1}^d \left(c_{jk}^- \langle v, a_k u \rangle + c_{jk}^+ \langle v, a_k^\dagger u \rangle \right)$$

for some complex constants c_{jk}^-, c_{jk}^+ . The left-hand and right-hand side make sense for $u, v \in \text{Dom}(N)$, therefore they can be extended by density and so

$$\begin{aligned} & \frac{d}{dt} \langle \xi', e^{-itH} a_j e^{itH} \xi \rangle \\ &= \sum_{k=1}^d \left(c_{jk}^- \langle \xi', e^{-itH} a_k e^{itH} \xi \rangle + c_{jk}^+ \langle \xi', e^{-itH} a_k^\dagger e^{itH} \xi \rangle \right) \end{aligned}$$

for all $\xi', \xi \in \text{Dom}(N)$. Considering the conjugate we find a differential equation for $\langle \xi', e^{-itH} a_j^\dagger e^{itH} \xi \rangle$ and so we get a linear system of $2d$ differential equations with constant coefficients. The solution of the system yields analytic functions H^-, H^+ as blocks of the exponential of a $2d \times 2d$ matrix. \square

Proof of Theorem 3.2. Let G_0 be the self-adjoint extension of $-\sum_{\ell=1}^d L_\ell^* L_\ell / 2$. By Proposition 4.12, for all $y \in \mathcal{N}(\mathcal{T})$ and all $v, u \in \text{Dom}(N)$, we have

$$\begin{aligned} & \langle G_0 v, yu \rangle + \sum_{\ell=1}^m \langle L_\ell v, yL_\ell u \rangle + \langle v, yG_0 u \rangle \\ &= -\frac{1}{2} \sum_{\ell=1}^m (\langle L_\ell^* L_\ell v, yu \rangle - 2 \langle L_\ell v, yL_\ell u \rangle + \langle v, yL_\ell^* L_\ell u \rangle) \\ &= -\frac{1}{2} \sum_{\ell=1}^m (\langle L_\ell^* y^* L_\ell v, u \rangle - 2 \langle L_\ell v, yL_\ell u \rangle + \langle v, L_\ell^* y L_\ell u \rangle) = 0 \end{aligned}$$

because $L_\ell^* y^*$ and $L_\ell^* y$ are extensions of $y^* L_\ell^*$ and $y L_\ell^*$ respectively, namely $\mathcal{L}(x) = i[H, x]$ (as a quadratic form).

Now, recalling that $\mathcal{N}(\mathcal{T})$ is \mathcal{T}_s -invariant by Proposition 3.1 1. for all $v, u \in \text{Dom}(N)$ also $e^{-i(t-s)H} v$ and $e^{-i(t-s)H} u$ belong to the domain of N by Lemma 4.13, we have

$$\frac{d}{ds} \langle e^{-i(t-s)H} v, \mathcal{T}_s(x) e^{-i(t-s)H} u \rangle = 0$$

which implies

$$\mathcal{T}_t(x) = e^{itH} x e^{-itH}.$$

From \mathcal{T}_t -invariance of $\mathcal{N}(\mathcal{T})$ it follows that also $e^{itH} x e^{-itH}$ belongs to the generalized commutant of the operators L_ℓ, L_ℓ^* ($\ell \geq 1$).

Since $\text{Dom}(N)$ is e^{itH} -invariant by Lemma 4.13, replacing $\xi \in \text{Dom}(N)$ by $e^{itH} \xi \in \text{Dom}(N)$ and left multiplying by e^{-itH} the identity $e^{itH} x e^{-itH} L_\ell \xi = L_\ell e^{itH} x e^{-itH} \xi$ becomes

$$x e^{-itH} L_\ell e^{itH} \xi = e^{-itH} L_\ell e^{itH} x \xi$$

Taking the scalar product with two exponential vectors and differentiating n times at $t = 0$ the identity

$$\langle v, x e^{-itH} L_\ell e^{itH} u \rangle = \langle e^{-itH} L_\ell^* e^{itH} v, x u \rangle$$

with $u, v \in \text{Dom}(N)$, we get

$$\langle v, x \delta_H^n(L_\ell) u \rangle = \langle \delta_H^n(L_\ell^*) v, x u \rangle.$$

Since iterated commutators $\delta_H^n(L_\ell)$ are first order polynomials in a_j, a_k^\dagger , this means that x belongs to the generalized commutant of $\delta_H^n(L_\ell)$. The same argument applies to generalized commutators of $\delta_H^n(L_\ell^*)$ for all $\ell \geq 1, n \geq 0$.

Conversely, if x belongs to the generalized commutant of operators $\delta_H^n(L_\ell), \delta_H^n(L_\ell^*)$ for all $\ell \geq 1, 0 \leq n \leq 2d - 1$, recall that each one of these generalized commutators is a first order polynomial in a_j, a_k^\dagger and so determines two vectors (coefficients of creation and annihilation operators) $\bar{v}, u \in \mathbb{C}^d$ and, eventually, a vector $[\bar{v}, u]^T \in \mathbb{C}^{2d}$. Let \mathcal{V}_n be the complex linear subspace of \mathbb{C}^{2d} determined by vectors in \mathbb{C}^{2d} corresponding to generalized commutators of order less or equal than n . Clearly, $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$ for all n and so the dimensions $\dim_{\mathbb{C}}(\mathcal{V}_n)$ form a non decreasing sequence of natural numbers bounded by $2d$. Moreover, if $\dim_{\mathbb{C}}(\mathcal{V}_n) = \dim_{\mathbb{C}}(\mathcal{V}_{n+1})$, then $\mathcal{V}_n = \mathcal{V}_{n+1}$ and so

$$\delta_H^{n+1}(L_\ell) = \sum_{m=0}^n \left(z_m \delta_H^{(m)}(L_\ell) + w_m \delta_H^{(m)}(L_\ell^*) \right) + \eta_n \mathbb{1},$$

$$\delta_H^{n+1}(L_\ell^*) = \sum_{m=0}^n (-1)^m \left(\bar{w}_m \delta_H^{(m)}(L_\ell) + \bar{z}_m \delta_H^{(m)}(L_\ell^*) \right) + \bar{\eta}_n \mathbb{1},$$

for some $z_1, \dots, z_n, w_1, \dots, w_n, \eta_n \in \mathbb{C}$. Iterating, it turns out that the linear part in creation and annihilation operators of $\delta_H^{n+m}(L_\ell)$ and $\delta_H^{n+m}(L_\ell^*)$ depends on vectors in \mathcal{V}_n for all $m \geq 0$. It follows that, starting from a value $n_0 \geq 1$ (corresponding to the zero order commutators L_ℓ and L_ℓ^*), the sequence of dimensions has to increase at least by 1 before reaching the maximum value. As a consequence, this is attained in at most $2d - 1$ steps.

Summarizing, if x belongs to the generalized commutant of operators $\delta_H^n(L_\ell), \delta_H^n(L_\ell^*)$ for all $\ell \geq 1, 0 \leq n \leq 2d - 1$, then it belongs to generalized commutant of these operators for all $n \geq 0$. By Lemma 4.14, we can consider the analytic function on \mathbb{R}

$$t \mapsto \langle \xi', x e^{-itH} L_\ell e^{itH} \xi \rangle - \langle e^{-itH} L_\ell^* e^{itH} \xi', x \xi \rangle$$

for all $\xi, \xi' \in D$. The n -th derivative at $t = 0$ is $(-i)^n$ times

$$\langle \xi', x \delta_H^n(L_\ell) \xi \rangle - \langle \delta_H^n(L_\ell^*) \xi', x \xi \rangle = 0$$

for all $n \geq 0$ because x belongs to the generalized commutant of operators $\delta_H^n(L_\ell)$. The same argument shows that x belongs to the generalized commutant of operators $\delta_H^n(L_\ell^*)$. Applying Theorem 4.1 of [14] (with $C = N$ and keeping in mind that $[G, C], [G^*, C]$ are second order polynomials in a_j, a_k^\dagger , therefore relatively bounded with respect to C whence with respect to $C^{3/2}$) it follows that $\mathcal{T}_t(x) = e^{itH} x e^{-itH}$.

The same conclusion holds for x^* and x^*x because they belong to the generalized commutant of operators $\delta_H^n(L_\ell), \delta_H^n(L_\ell^*)$. Therefore $x \in \mathcal{N}(\mathcal{T})$ and the proof is complete. □

Appendix C: Proof of Proposition 3.6

Proof. Statement 2. readily follows by noticing that if $z \in M_1$ is such that $\mathfrak{S}\langle z, z_1 \rangle = 0$ for all $z_1 \in M_1$, then $z \in M_1 \cap M_1'$. Therefore $\mathfrak{S}\langle \cdot, \cdot \rangle$ is non-degenerate when restricted to M_1 if and only if $M_1 \cap M_1' = \{0\}$.

We now prove the first one on the existence of the symplectomorphism for M . The first step in this proof is an adaptation of the Gram-Schmidt procedure to symplectic spaces. Consider $z_1 \in M$ with $z_1 \neq 0$ and observe that there exists $z \in M$ such that $\mathfrak{S}\langle z_1, z \rangle \neq 0$, otherwise the symplectic form would be degenerate on M . We can now set $z_2 = z/\mathfrak{S}\langle z_1, z \rangle$, so that $\mathfrak{S}\langle z_1, z_2 \rangle = 1$. Let $M_1 = \text{Lin}_{\mathbb{R}}\{z_1, z_2\}$ we now show that $\dim_{\mathbb{R}} M_1 = 2$ and

$$M = M_1 \oplus M_1',$$

where both M_1 and M_1' are symplectic spaces. Clearly if z_1, z_2 were linearly dependent we would have $z_1 = sz_2$ for some $s \in \mathbb{R}$ and then $\mathfrak{S}\langle z_1, z_2 \rangle = 0$ which contradicts the construction of z_2 . Again since $\mathfrak{S}\langle z_1, z_2 \rangle \neq 0$ we have $M_1 \cap M_1' = \{0\}$ and M_1 is a symplectic subspace for what we proved at the beginning. Moreover if $z_1 \in M_1$ we can write

$$z = (z + \mathfrak{S}\langle z, z_1 \rangle z_2 - \mathfrak{S}\langle z, z_2 \rangle z_1) + (\mathfrak{S}\langle z, z_2 \rangle z_1 - \mathfrak{S}\langle z, z_1 \rangle z_2),$$

where it holds

$$z + \mathfrak{S}\langle z, z_1 \rangle z_2 - \mathfrak{S}\langle z, z_2 \rangle z_1 \in M_1', \quad \mathfrak{S}\langle z, z_2 \rangle z_1 - \mathfrak{S}\langle z, z_1 \rangle z_2 \in M_1,$$

hence we have proved $M = M_1 \oplus M_1'$. Eventually M_1' is a symplectic space since it holds $M_1'' = M_1$ and $M_1'' \cap M_1' = \{0\}$. Note also that $\dim_{\mathbb{R}} M_1' \neq 1$ otherwise the symplectic form would be degenerate on it.

We can now repeat the same reasoning starting with the symplectic space M_1' in order to obtain $z_3, z_4 \in M_1'$ such that $\mathfrak{S}\langle z_3, z_4 \rangle = 1$ and setting $M_2 = \text{Lin}_{\mathbb{R}}\{z_3, z_4\}$ we have

$$M = M_1 \oplus M_2 \oplus M_2',$$

where M_1, M_2, M_2' are symplectic spaces with $\dim_{\mathbb{R}} M_j = 2$ and $\dim_{\mathbb{R}} M_2' \neq 1$. Notice that they are also pairwise symplectically orthogonal, since both $M_2, M_2' \subset M_1'$. Since the remainder space M_j' has always dimension different from 1 we can iterate this process until we get $M_j' = \{0\}$. When the procedure stops we have a sequence M_1, \dots, M_{d_1} of mutually (symplectically) orthogonal symplectic spaces, with $M_j = \text{Lin}_{\mathbb{R}}\{z_{2j}, z_{2j+1}\}$, $\mathfrak{S}\langle z_{2j}, z_{2j+1} \rangle = 1$. Clearly $2d_1 = \dim_{\mathbb{R}} M$ and this concludes the first step of the proof.

In order to conclude the proof of this point is sufficient to construct the symplectomorphism via

$$Bz_{2j} = e_j, \quad Bz_{2j+1} = ie_j.$$

Eventually for the symplectomorphism of $M_1 \subset M$ an isotropic subset, consider $\{z_1, \dots, z_{d_1}\}$ a real linear basis of M_1 with $d_1 = \dim_{\mathbb{R}} M_1$. Since M_1 is isotropic we have

$$\mathfrak{S}\langle z_j, z_k \rangle = 0 \quad \forall j, k = 1, \dots, d_1.$$

This proves 2. In order to prove 3. it suffices to define

$$Bz_j = e_j,$$

to get a symplectomorphism B . □

References

- [1] Agredo, J., Fagnola, F., Poletti, D.: Gaussian Quantum Markov Semigroups on a One-Mode Fock Space: Irreducibility and Normal Invariant States. *Open Sys. Information Dyn.* **28**(1), 2150001 (2021)
- [2] Agredo, J., Fagnola, F., Rebolledo, R.: Decoherence free subspaces of a quantum Markov semigroup. *J. Math. Phys.* **55**, 112201 (2014)
- [3] Araki, H.: A Lattice of von Neumann Algebras Associated with the Quantum Theory of a Free Bose Field. *J. Math. Phys.* **4**, 1343–1362 (1963)
- [4] Blanchard, Ph., Olkiewicz, R.: Decoherence induced transition from quantum to classical dynamics. *Rev. Math. Phys.* **15**(3), 217–243 (2003)
- [5] Bratteli, O., Robinson, D.W.: *Operator Algebras and Quantum Statistical Mechanics I*. Springer-Verlag, Berlin (1979). (**Second Printing 2002**)
- [6] Cannas da Silva, A.: *Lectures on Symplectic Geometry*. Corrected 2nd printing 2008. Springer, Berlin Heidelberg (2008)
- [7] Carbone, R., Sasso, E., Umanità, V.: Environment induced decoherence for Markovian evolutions. *J. Math. Phys.* **56**, 092704 (2015)
- [8] Cattaneo, M., Giorgi, G.L., Maniscalco, S., Zambrini, R.: Symmetry and block structure of the Liouvillian superoperator in partial secular approximation. *Phys. Rev. A* **101**, 042108 (2020)
- [9] Chebotarev, A.M., Fagnola, F.: Sufficient Conditions for Conservativity of Minimal Quantum Dynamical Semigroups. *J. Funct. Anal.* **153**(3), 382–404 (1998)
- [10] Crismale, V., Fidaleo, F.: Symmetries and ergodic properties in quantum probability. *Colloq. Math.* **149**, 1–20 (2017)
- [11] Demoen, B., Vanheuverzwijn, P., Verbeure, A.: Completely positive maps on the CCR-algebra. *Lett. Math. Phys.* **2**, 161–166 (1977)
- [12] Dereziński, J.: Bosonic quadratic Hamiltonians. *J. Math. Phys.* **58**, 121101 (2017)
- [13] Deschamps, J., Fagnola, F., Sasso, E., Umanità, V.: Structure of uniformly continuous quantum Markov semigroups. *Rev. Math. Phys.* **28**(1), 1650003 (2016)
- [14] Dhahri, A., Fagnola, F., Rebolledo, R.: The decoherence-free subalgebra of a quantum Markov semigroup with unbounded generator, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **13**(3), 413–433 (2010)
- [15] Evans, D.E.: Irreducible quantum dynamical semigroups. *Commun. Math. Phys.* **54**, 293–297 (1977)
- [16] Fagnola, F.: Quantum Markov Semigroups and Quantum Markov Flows. *Proyecciones* **18**(3), 1–144 (1999)
- [17] Fagnola, F.: A simple singular quantum Markov semigroup. *Proceedings of the Third International Workshop Stochastic Analysis and Mathematical Physics ANESTOC '98* Birkäuser p. 73–88 (2000)

- [18] Fagnola, F., Sasso, E., Umanità, V.: The role of the atomic decoherence-free subalgebra in the study of quantum Markov semigroups. *J. Math. Phys.* **60**, 072703 (2019)
- [19] Fagnola, F., Gough, J.E., Nurdin, H.I., Viola, L.: Mathematical models of Markovian dephasing. *J. Phys. A* **52**(38), 385301 (2019)
- [20] Gough, J.E., James, M.R., Nurdin, H.I.: Squeezing components in linear quantum feedback networks. *Phys. Rev. A* **81**, 023804 (2010)
- [21] Hislop, P.D.: A simple proof of duality for local algebras in free quantum field theory. *J. Math. Phys.* **27**, 2542–2550 (1986)
- [22] Isar, A.: Decoherence and asymptotic entanglement in open quantum dynamics. *J. Russ. Laser Res.* **28**(5), 439–452 (2007)
- [23] Jorgensen, P.E.T.: Approximately Reducing Subspaces for Unbounded Linear Operators. *J. Funct. Anal.* **23**, 392–414 (1976)
- [24] Kato, T.: Perturbation theory for linear operators. Corrected printing of the second edition. Springer-Verlag, Berlin, Heidelberg, New York (1980)
- [25] Leyland, P., Roberts, J., Testard, D.: Duality for free quantum Fields, Centre de Physique Théorique Marseille, Report CPT-78/P-1016 (1978). <https://inspirehep.net/literature/132161>
- [26] Lidar, D.A.: Review of decoherence free subspaces, noiseless subsystems, and dynamical decoupling. *Adv. Chem. Phys* **154**, 295–354 (2014)
- [27] Lidar, D.A., Chuang, I.L., Whaley, K.B.: Decoherence-free subspaces for quantum computation. *Phys. Rev. Lett.* **81**, 2594 (1998)
- [28] Parthasarathy, K.R.: An introduction to Quantum Stochastic Calculus. Birkhauser, Basel (1992)
- [29] Poletti, D.: Characterization of Gaussian Quantum Markov Semigroups. In: Proceedings QP41 (to appear)
- [30] Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Fourier Analysis Self-Adjointness Academic Press, San Diego, II (1975)
- [31] Teretenkov, A.E.: Irreversible quantum evolution with quadratic generator. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **22**(4), 1930001 (2019)
- [32] Vanheuverzwijn, P.: Generators for quasi-free completely positive semigroups. *Ann. Inst. H. Poincaré Sect. A (N.S.)* **29**, 123–138 (1978)

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